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040: : |a MiU |c MiU

100:1 : |a Wells, Webster, |d 1851-1916.

245:00: |a New plane geometry, |c by Webster Wells.

260: : |a Boston, |b D.C. Heath & co., |c 1909.

300/1: : |a vii p., 1 L., 298 p. |b viii col. pl., diagrs. |c 18 cm.

500/1: : |a "In the prepatation of this text the author acknowledges joint authorship with Robert L. Short."--Pref.

500/2: : |a "New solid geometry" (1 L., 175-298 p.) also issued separately.

650/1: 0: |a Geometry

700/1:1 : |a Short, Robert Louis, |d 1866-

998: : |c RAS |s 9124

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NEW
PLANE AND SOLID
GEOMETRY

BY

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BOSTON, U.S.A.

D. C. HEATH & CO., PUBLISHERS

1909

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PREFACE

IN the preparation of this text the author acknowledges joint authorship with Robert L. Short.

Geometry is approached from the constructive side, all methods of construction needed for drawing any figure in Books I or II being given in the introduction. In cases where a geometric principle is used in any construction, a note at the end tells where the principle is proved. In all figures in the early portions of Book I, the construction lines and arcs are given; afterwards these are dispensed with.

In Props. II and III, Book I, and in other places, colored diagrams are given in addition to the regular figures, in which the equal parts in the given triangles are represented by *lines of the same color*; this scheme will be found of great assistance to the pupil in the earlier portions of the work.

Below each figure is a paragraph in smaller type, giving full directions for the construction of the diagram in accordance with the statement of the theorem. This gives the pupil a familiarity with the figure, and what in it is given, and what to be proved, that is of great value, and lessens the tendency to memorize. Many figures are omitted, but complete directions for their construction are given in each case.

In all figures given in connection with theorems and problems, given and required lines are made *heavy*.

Attention is invited to the order of theorems in Book I; in this case, the pupil begins with the easier proofs. From the start the student has practice in representing angles and lines by small letters.

Only the outline of the proof is given after Book II, except in the more difficult demonstrations. In this outline work the pupil has explicit directions, but develops the demonstration himself. In all portions of the work proofs are omitted in

cases where they should present no difficulty ; usually, in such cases, hints are given as to the method of demonstration.

The numbering of the steps of the proof makes them more easy of reference.

In Book I, and in the first part of Book VI, the authority for each statement of the proof will be found directly below, in smaller type, enclosed in brackets, with the number of the section where it is to be found. In other portions of the work only the section number is given, and in some cases only an interrogation point. In all such cases the pupil should be required to give the authority as fully as if it were actually printed on the page.

No principles are given as immediate consequences of theorems except such as actually belong in this category. Separate propositions are made for all truths which are not immediate consequences of preceding theorems.

The originals are new, and very largely of a practical nature. They are not too difficult, will make the pupil think, and are more than five hundred in number. The smaller number is compensated for by the fact that the pupil has to do some original work in almost every proof after Book II. Exercises coming under Book I will be found scattered through all the following Books of the Plane Geometry ; and a similar remark applies to exercises under Book II, Book III, etc.

At the end of Book I will be found a list of principles proved, which will be of great assistance in solving originals. A similar list in regard to similar triangles is given in § 265.

The authors thought that it would be an improvement to put all work under Polyedrons, including their measurement, in Book VII, devoting Book VIII to the Cylinder and Cone and Book IX to the Sphere and its measurement.

The authors wish to thank the many teachers whose advice and criticism have been useful in preparing a treatise that should stand the test of class-room work. They are also indebted to Mr. C. W. Sutton for a part of the exercises.

WEBSTER WELLS.

BOSTON, 1908.

SUGGESTIONS TO TEACHERS

IN studying the opening propositions, in Geometry, beginners have difficulty in fixing clearly in mind just what parts, in the figure, are given. To aid this, in Props. II and III, Book I, and in other places, diagrams are given in addition to the regular cuts, in which the equal given parts are printed in the same color.

This color-scheme may be advantageously followed in the class-room, in connection with all figures in the earlier portions of Book I. If colored crayons are not available, the beginner may designate equal lines by the marks $'$, $''$, $'''$, etc., and equal angles by single, double, or triple *arcs*.

In solving the non-numerical exercises, while it is not practicable to give very much assistance to the pupil, the following suggestions may be found of service:—

1. *Draw an accurate figure, showing all given and required lines.*

This sometimes suggests the method of proof.

2. *Be sure that the figure is the most general one allowable.*

Thus, in exercises relating to triangles, do not draw them right, isosceles, or equilateral, unless the exercise calls for a right, isosceles, or equilateral triangle; in exercises relating to quadrilaterals, do not draw them with two sides equal or parallel, unless the exercise calls for such a construction.

3. *Write down carefully what is given, and what is to be proved.*

4. *Look up all previous theorems which may possibly have a bearing on what is to be proved.*

Thus, if two lines are to be proved equal, refer to all previous theorems regarding equal lines. (Compare § 141.)

In solving exercises in construction, it is advantageous to regard the problem as solved, and draw the given and required lines. Studying the relations between these will frequently suggest the method of construction to be employed.

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SOLID GEOMETRY

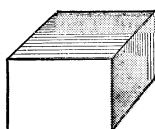
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NEW PLANE GEOMETRY

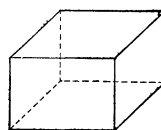
GEOMETRY



INTRODUCTORY



A material body



A geometrical solid

1. A *material body*, as, for example, a block of wood, occupies a *limited* portion of space.

The boundary which separates such a body from surrounding space is called the *surface* of the body.

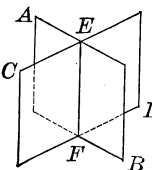
If the material which composes such a body could be conceived as taken away from it, *without altering the form of the bounding surface*, we should have a *portion of space*, with the same bounding surface as the material body.

We call this portion of space a *geometrical solid*, or simply a *solid*.

We call the surface which bounds it a *geometrical surface*, or simply a *surface*; it is also called the *surface of the solid*.

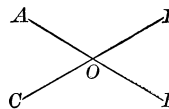
2. If two surfaces intersect each other, we call that which is common to both a *geometrical line*, or simply a *line*.

Thus, if surfaces AB and CD cut each other, their common part, EF , is a line.



3. If two lines intersect each other, we call that which is common to both a *geometrical point*, or simply a *point*.

Thus, if lines AB and CD cut each other, their common part, O , is a point.



4. A solid has *extension in every direction*; but this is not the case with surfaces and lines.

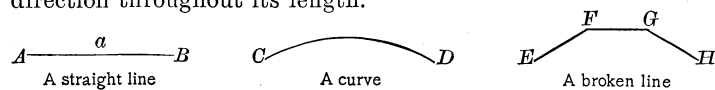
A point has extension in *no* direction.

We may conceive a surface as existing independently in space, without reference to the solid whose boundary it forms.

In like manner, we may conceive of lines and points as existing independently in space.

A line is produced by the motion of a moving point, a surface by the motion of a moving line, and a solid by the motion of a moving surface.

5. We define a *straight line* as a line which has the same direction throughout its length.



A straight line is designated by two letters anywhere upon its length, or by a single letter; thus, the straight line in the figure may be designated either AB or a .

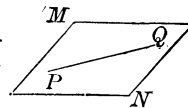
We define a *curve* as a line no portion of which is straight; as CD .

We define a *broken line* as a line which is composed of different successive straight lines; as $EFGH$.

The word "*line*," without qualification, will be used hereafter as signifying a *straight line*.

6. We define a *plane* as a surface such that if a straight line be drawn between any two of its points, it lies entirely in the surface.

Thus, if P and Q are any two points in surface MN , and the straight line PQ lies entirely in the surface, then MN is a plane.



7. We may conceive a straight line as being of unlimited length; we may also conceive a plane as being of unlimited extent in regard to length and breadth.

8. We define a *geometrical figure* as any combination of points, lines, surfaces, and solids.

We define a *plane figure* as a figure formed by points and lines all lying in the same plane.

A figure is called *rectilinear* when it is composed of straight lines only.

A straight line is sometimes called a *right line*.

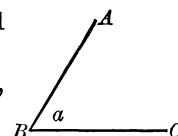
9. *Geometry* treats of the properties, construction, and measurement of geometrical figures.

Plane Geometry treats of plane figures only.

Solid Geometry treats of figures which are not plane.

10. We define an *angle* as the figure formed by two straight lines drawn from a point.

The point is called the *vertex* of the angle, and the straight lines its *sides*.



11. If there is but one angle at a given vertex, we designate it by the letter at the vertex.

But if two or more angles have the same vertex, we avoid ambiguity by naming also a letter on each side, reading the vertex letter between the others.

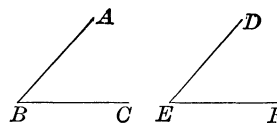
Thus, we should read the angle of § 10 “angle *B*”; but if there were other angles at the same vertex, we should read it either *ABC* or *CBA*.

Another way of designating an angle is by means of a letter placed between its sides; thus, we may read the angle of § 10 “angle *a*.”

12. Two figures are said to be *equal* when one can be applied to the other so that they shall coincide throughout.

To prove two *angles* equal, we do not consider the lengths of their sides.

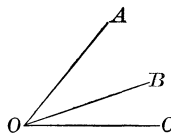
Thus, if angle *ABC* can be applied to angle *DEF* in such a manner that point *B* shall fall on point *E*, side *AB* on side *DE*, and side *BC* on side *EF*, the angles are equal, even if side *AB* is not equal in length to side *DE*, and *BC* to *EF*.



13. We call two angles *adjacent* when they have the same vertex, and a common side between them; as $\angle AOB$ and $\angle BOC$.

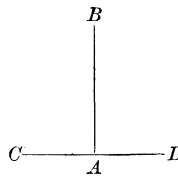
We call angle $\angle AOC$ the *sum* of angles $\angle AOB$ and $\angle BOC$.

We also regard angle $\angle AOC$ as greater than angle $\angle BOC$, and angle $\angle BOC$ as less than angle $\angle AOC$.



14. If from a point in a straight line a line be drawn in such a way as to make the adjacent angles equal, each of the adjacent angles is called a *right angle*, and the lines are said to be *perpendicular* to each other.

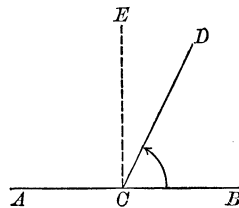
Thus, if from point A in line CD line AB be drawn in such a way as to make angles $\angle BAC$ and $\angle BAD$ equal, each of these angles is a right angle, and AB and CD are perpendicular to each other.



15. Let C be any point in straight line AB ; and let straight line CD be drawn in such a way as to make angle $\angle BCD$ less than angle $\angle ACD$.

Let line CD be turned about point C as a pivot towards the position CA .

Then, angle $\angle BCD$ will constantly increase, and angle $\angle ACD$ will constantly diminish; and there must be some position of CD where these angles are equal, and there can evidently be but one such position.



If CE is this position, by the definition of § 14, CE is perpendicular to AB at C .

Then, *at a given point in a straight line, a perpendicular to the line can be drawn, and but one.*

The following are immediate consequences of § 15:

16. *Any two right angles are equal.*

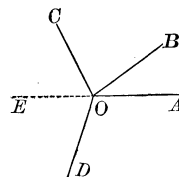
17. *If two adjacent angles have their exterior sides in the same straight line, their sum is equal to two right angles.*

For in figure of § 15, the sum of angles ACD and BCD equals the sum of angles ACE and BCE , or two right angles.

18. *The sum of all the angles on the same side of a straight line at a given point is equal to two right angles.*

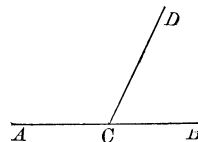
19. *The sum of all the angles about a point in a plane is equal to four right angles.*

For if a side of any angle, as OA , be extended to E , the sum of the angles on either side of straight line AE is, by § 18, equal to two right angles.



20. *If the sum of two adjacent angles is equal to two right angles, their exterior sides lie in the same straight line.*

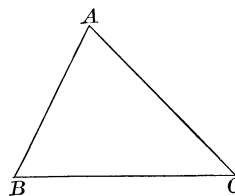
If the sum of angles ACD and BCD is two right angles, side AC if extended through C must coincide with CB ; for if it did not the sum of the angles would be less or greater than 180° (§ 17).



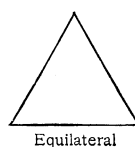
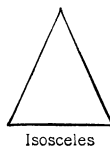
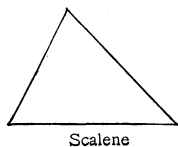
21. We define a *triangle* as a portion of a plane bounded by three straight lines; as ABC .

We call the lines AB , BC , and CA the *sides* of the triangle; and their intersections, A , B , and C , the *vertices*.

We define the *angles* of the triangle as the angles CAB , ABC , and BCA , between the adjacent sides.

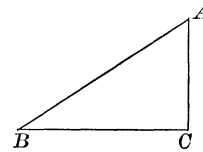


22. A triangle is called *scalene* when no two sides are equal; *isosceles* when two sides are equal; *equilateral* when all its sides are equal; *equiangular* when all its angles are equal.



A *right triangle* is a triangle which has a right angle; as ABC , which has a right angle at C .

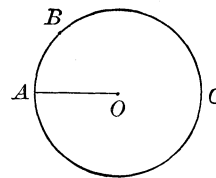
The side AB opposite the right angle is called the *hypotenuse*, and the other sides, AC and BC , the *legs*.



23. We define a *circle* as a portion of a plane bounded by a curve, called the *circumference*, all points of which are equally distant from a point within called the *centre*.

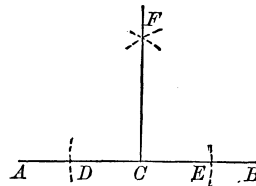
An *arc* is any portion of the circumference; as AB .

A *radius* is a straight line drawn from the centre to the circumference; as OA .



CONSTRUCTIONS

24. At a given point in a straight line, to draw a perpendicular to the line.



Let it be required to draw a line perpendicular to the straight line AB , at the point C .

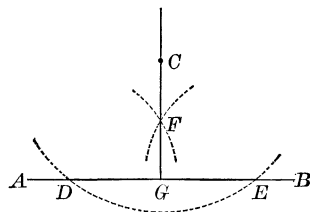
With C as centre, and any straight line less than AC as radius, describe arcs intersecting AC at D , and BC at E .

With points D and E as centres, and any radius, describe arcs intersecting at F .

Then, the straight line drawn from C through F will be perpendicular to AB at C .

The reason for the above construction will be found in § 58.

25. *From a given point without a straight line to draw a perpendicular to the line.*



Let it be required to draw through any point C without the straight line AB a line perpendicular to AB .

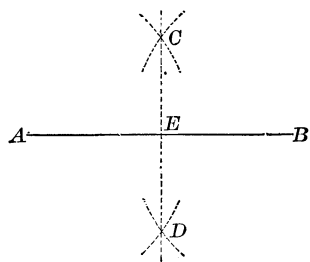
With C as a centre, and any radius, describe an arc cutting AB at points D and E .

With points D and E as centres, and another radius, describe arcs intersecting at F .

Then, the straight line CG drawn from C through F will be perpendicular to AB .

The reason for the construction will be found in § 58.

26. *To bisect a given straight line.*



Let it be required to divide the straight line AB into two equal parts.

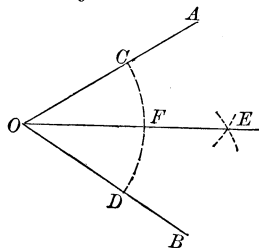
With points A and B as centres, and with the same radius, describe arcs intersecting at C and D .

Draw straight line CD intersecting AB at E .

Then, E is the middle point of AB .

The reason for the construction will be found in § 58.

27. *To bisect a given angle.*



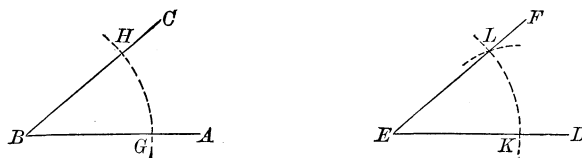
Let it be required to bisect angle AOB .

With O as a centre, and any radius, describe an arc intersecting OA at C and OB at D ; with C and D as centres, and the same radius, describe arcs intersecting at E .

Then, the straight line OE will bisect angle AOB .

The reason for the construction will be found in § 58.

28. *With a given vertex, and a given side, to construct an angle equal to a given angle.*



Let it be required to construct with E as the vertex, and ED as a side, an angle equal to angle ABC .

With B as a centre, and any radius, describe an arc intersecting AB at G and BC at H .

With E as a centre, and BG as a radius, describe an arc intersecting ED at K .

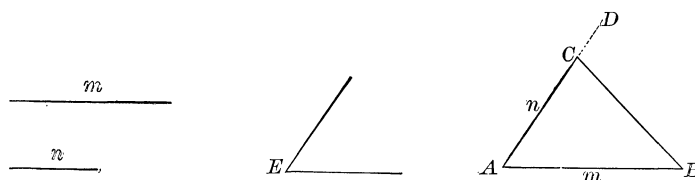
With K as a centre, and the distance from G to H as a radius, describe an arc intersecting the former arc at L .

Draw straight line ELF .

Then, angle DEF will equal angle ABC .

The reason for the construction will be found in § 53.

29. *Given two sides and the included angle of a triangle, to construct the triangle.*



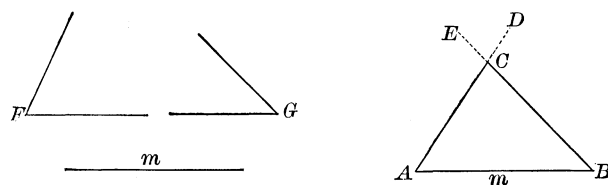
Let it be required to construct the triangle having for two of its sides the straight lines m and n , and their included angle equal to angle E .

Draw line AB equal to m , and construct angle BAD equal to angle E (§ 28).

On AD take AC equal to n , and draw straight line BC .

Then, ABC is the required triangle.

30. *Given a side and two adjacent angles of a triangle, to construct the triangle.*



Let it be required to construct the triangle having for a side the straight line m , and its adjacent angles equal to angles F and G .

Draw line AB equal to m , and construct angle BAD equal to angle F (§ 28).

Draw line BE , making angle ABE equal to angle G , intersecting AD at C .

Then, ABC is the required triangle.

31. *Given the three sides of a triangle, to construct the triangle.*



Let it be required to construct the triangle having for its sides the straight lines m , n , and p .

Take the straight line AB equal to m .

With A as a centre, and n as a radius, describe an arc.

With B as a centre, and p as a radius, describe an arc intersecting the former arc at C .

Then, ABC is the required triangle.

32. We define an *Axiom* as a truth which is assumed without proof as being self-evident.

We define a *Theorem* as a truth requiring proof.

We define a *Problem* as a question proposed for solution.

A *Proposition* is a general term for a theorem or a problem.

A *Postulate* assumes that a certain problem can be solved.

A *Corollary* is a truth which is an immediate consequence of the proposition which it follows.

An *Hypothesis* is a supposition, made either in the statement or the proof of a proposition.

33. Postulates.

We assume that the following problems can be solved :

1. A straight line can be drawn between any two points.
2. A straight line can be extended indefinitely in either direction.

34. Axioms.

We assume the following as true :

1. *Things which are equal to the same thing, or to equal things, are equal to each other.*
2. *If equals be added to equals, the sums will be equal.*

3. *If equals be subtracted from equals, the remainders will be equal.*

4. *If equals be multiplied by equals, the products will be equal.*

5. *If equals be divided by equals, the quotients will be equal.*

6. *But one straight line can be drawn between two points.*

7. *A straight line is the shortest line between two points.*

8. *The whole is equal to the sum of all its parts.*

9. *The whole is greater than any of its parts.*

35. Since but one straight line can be drawn between two points, a straight line is said to be *determined* by any two of its points.

36. Symbols and Abbreviations.

The following symbols are used in the work :

+, plus.	$\triangle$, triangle.
−, minus.	$\triangle$, triangles.
$\times$, multiplied by.	$\perp$, perpendicular, is perpendicular to.
=, equals.	$\perp$, perpendiculars.
$\simeq$, equivalent, is equivalent to.	$\parallel$, parallel, is parallel to.
$>$, is greater than.	$\parallel$, parallels.
$<$, is less than.	$\square$, parallelogram.
$\therefore$, therefore.	$\square$, parallelograms.
$\angle$, angle.	$\odot$, circle.
$\sphericalangle$, angles.	$\odot$, circles.

The following abbreviations are used:

Ax., Axiom.	Sup., Supplementary.
Def., Definition.	Alt., Alternate.
Hyp., Hypothesis.	Int., Interior.
Cons., Construction.	Ext., Exterior.
Rt., Right.	Corresp., Corresponding.
Str., Straight.	Rect., Rectangle, rectangular.
Adj., Adjacent.	

Book I

RECTILINEAR FIGURES

DEFINITIONS

37. We define an *acute* angle as an angle less than a right angle; as ABC .

We define an *obtuse* angle as an angle greater than a right angle; as DEF .

Acute and obtuse angles are called *oblique* angles; and intersecting lines which are not perpendicular, are said to be oblique to each other.

We call two angles *vertical* when the sides of one are the prolongations of the sides of the other; as AEC and BED .

38. If angles AOB, BOC, COD, DOE , are all equal, we say that angle AOB is contained *four times* in angle AOE ; and similarly for any number of equal parts of angle AOE .

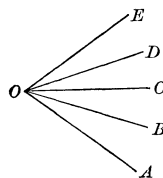
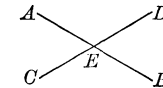
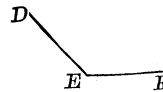
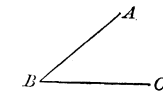
39. We *measure* an angle by finding how many times it contains another angle taken as the unit of measure.

The usual unit of measure for angles is the *degree*, which is the ninetieth part of a right angle.

To express fractional parts of the unit, we divide the degree into sixty equal parts, called *minutes*, and the minute into sixty equal parts, called *seconds*.

We represent degrees, minutes, and seconds by the symbols $^{\circ}, ', ''$, respectively.

40. If the sum of two angles is a right angle, or 90° , we call one the *complement* of the other; if their sum is two right angles, or 180° , we call one the *supplement* of the other.



Thus, the complement of an angle of 34° is $90^\circ - 34^\circ$, or 56° ; the supplement of an angle of 34° is $180^\circ - 34^\circ$, or 146° .

Two angles which are complements of each other are called *complementary*; and two angles which are supplements of each other are called *supplementary*.

41. It follows from the above that:

1. *The complements of equal angles are equal.*
2. *The supplements of equal angles are equal.*

42. Since $\angle ACD, BCD$ (Fig., § 15), are *supplementary* (§ 40); the principle of § 17 may be stated as follows:

If two adjacent angles have their exterior sides in the same straight line, they are supplementary.

Such angles are called *supplementary-adjacent*.

Ex. 1. How many degrees are there in the complement of 43° ? of 85° ?

Ex. 2. How many degrees are there in the supplement of 17° ? of 162° ?

Ex. 3. Find the supplement of the complement of 75° ; of $22^\circ 30'$; of $45^\circ 12' 18''$; of A° .

Ex. 4. Find the complement of the supplement of 178° ; of 144° ; of $125^\circ 14' 15''$; of B° .

Ex. 5. Given the sum and difference of two lines, to find the lines. (Bisect the sum, also bisect the difference.)

Ex. 6. The sum of the lengths of two given lines is 20 inches, and their difference is 4 inches; find the lines. Make a diagram.

43. Note. The demonstration of a geometric truth consists of three parts:

1. *The statement of what is given in the figure.*
2. *The statement of what is to be proved.*
3. *The proof, depending upon previous theorems, axioms, or definitions.*

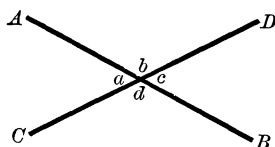
We shall mark these three divisions of the demonstration by the words **Given**, **To Prove**, and **Proof**, respectively.

We shall use the symbols and abbreviations of § 36, and also number the successive steps of the proof.

In every proof in Book I, where a preceding theorem is given as the authority for any statement in the proof, it will be found directly after the statement, in smaller type, enclosed in brackets.

PROP. I. THEOREM

44. *If two straight lines intersect, the vertical angles are equal.*



Draw intersecting str. lines AB and CD forming vertical $\angle a$ and c , and also b and d . We then have :

Given intersecting str. lines AB and CD , forming vertical $\angle a$ and c , and also b and d ,

To Prove $\angle a = \angle c$ and $\angle b = \angle d$.

Proof. 1. $\angle a + \angle b = 180^\circ$.

[If two adj. $\angle$ s have their ext. sides in the same str. line, their sum is equal to two rt. $\angle$ s.] (§ 17)

2. Also, $\angle b + \angle c = 180^\circ$.

3. Then, $\angle a + \angle b = \angle b + \angle c$.

[Things which are equal to the same thing, or to equal things, are equal to each other.] (Ax. 1)

4. Subtracting $\angle b$ from the equals $\angle a + \angle b$ and $\angle b + \angle c$,
 $\angle a = \angle c$.

[If equals be subtracted from equals, the remainders will be equal.] (Ax. 3)

5. In like manner, $\angle b = \angle d$.

45. The enunciation of every theorem consists essentially of two parts: the *Hypothesis*, and the *Conclusion*.

Thus, we may enunciate Prop. I as follows :

Hypothesis. If two straight lines intersect,

Conclusion. The vertical angles are equal.

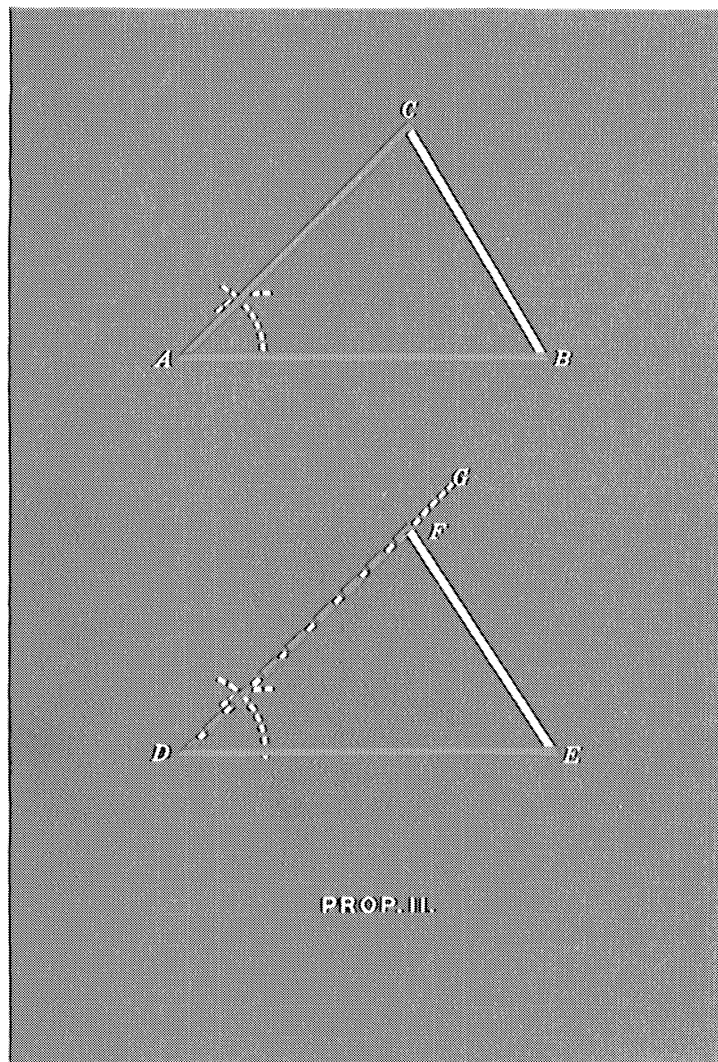
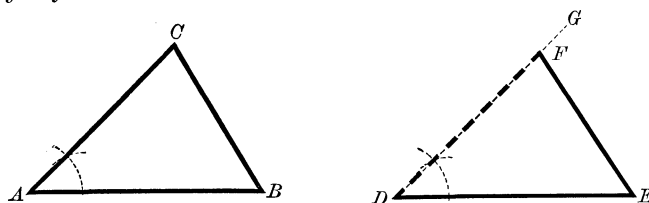


PLATE I

PROP. II. THEOREM

46. *Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*



Draw any $\triangle ABC$: draw line DE equal to AB : construct $\angle EDG$ equal to $\angle A$, the constructed side being DG : on DG take DF equal to AC : draw line FE . We now have:

Given, in $\triangle ABC$ and DEF ,

$$AB = DE, AC = DF, \text{ and } \angle A = \angle D.$$

To Prove $\triangle ABC = \triangle DEF$.

Proof. 1. Superpose $\triangle ABC$ upon $\triangle DEF$ in such a way that $\angle A$ shall coincide with its equal $\angle D$; side AB falling on side DE , and side AC on side DF .

2. Since $AB = DE$, point B will fall on point E .

3. Since $AC = DF$, point C will fall on point F .

4. Then, side BC will fall on side EF .

[But one str. line can be drawn between two points.] (Ax. 6)

5. Therefore, the $\triangle$ coincide throughout, and are equal.

47. Since ABC and DEF coincide throughout, we have

$$\angle B = \angle E, \angle C = \angle F, \text{ and } BC = EF.$$

48. Note. In equal figures, lines or angles which are similarly placed are called *homologous*.

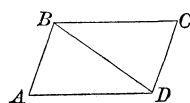
Thus, in the figure of Prop. II, $\angle A$ is homologous to $\angle D$; AB is homologous to DE ; etc.

It follows from the above that

In equal figures, the homologous parts are equal.

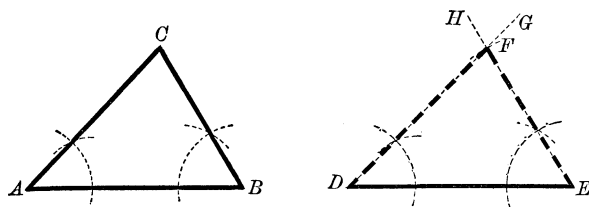
In equal triangles, the equal sides lie opposite the equal angles.

Ex. 7. If, in triangles ABD and BCD , sides AB and BD and angle ABD are equal respectively to sides CD , BD , and angle BDC , what other sides and angles of the triangles are equal?



PROP. III. THEOREM

49. *Two triangles are equal when a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other.*



Draw any $\triangle ABC$: draw line DE equal to AB : at D construct $\angle EDG$ equal to $\angle A$: at E construct $\angle DEH$ equal to $\angle B$. Let F be point of intersection of DG and EH . We now have:

Given, in $\triangle ABC$ and DEF ,

$$AB = DE, \angle A = \angle D, \text{ and } \angle B = \angle E.$$

To Prove $\triangle ABC = \triangle DEF$.

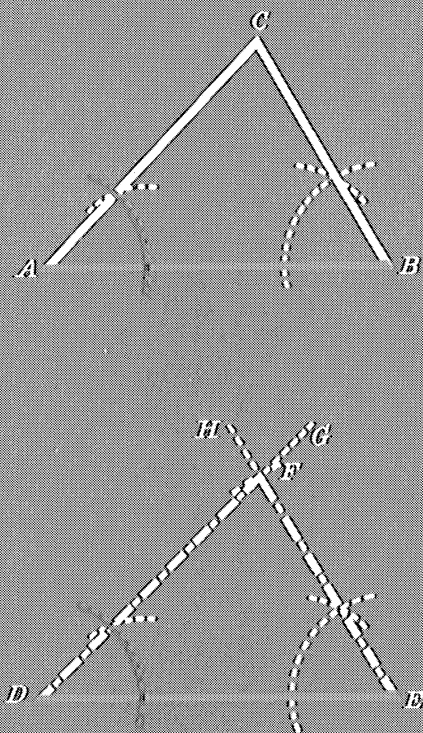
Proof. 1. Superpose $\triangle ABC$ upon $\triangle DEF$ in such a way that side AB shall coincide with its equal DE ; point A falling on point D , and point B on point E .

2. Since $\angle A = \angle D$, side AC will fall on side DF , and point C will fall somewhere on DF .

3. Since $\angle B = \angle E$, side BC will fall on side EF , and point C will fall somewhere on EF .

4. Since point C falls at the same time on DF and EF , it must fall at their intersection, F .

5. Then, the $\triangle$ coincide throughout and are equal.

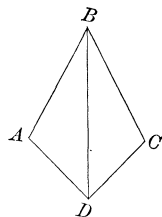


PROP. III.

PLATE II

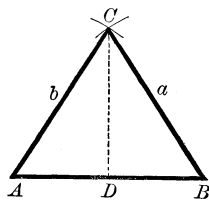
Ex. 8. What other sides and angles in the triangles of Prop. III are equal?

Ex. 9. If, in triangles ABD and BCD , side BD and angles ABD and ADB are equal respectively to side BD and angles CBD and CDB , what other sides and angles of the triangles are equal?



PROP. IV. THEOREM

50. In an isosceles triangle, the angles opposite the equal sides are equal.



Draw line AB . With any radius greater than $\frac{1}{2}AB$ and with A as a centre draw an arc; with the same radius and B as a centre draw an arc intersecting the first arc at C ; draw lines $AC(b)$ and $BC(a)$. We then have:

Given a and b , the equal sides of isosceles $\triangle ABC$.

To Prove $\angle A = \angle B$.

Proof. 1. Draw str. line CD bisecting $\angle ACB$, meeting AB at D .

2. In $\triangle ACD$ and BCD , $CD = CD$.

3. By hyp., $a = b$, and $\angle ACD = \angle BCD$.

4. Then, $\triangle ACD = \triangle BCD$.

[Two $\triangle$ are equal when the two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 46)

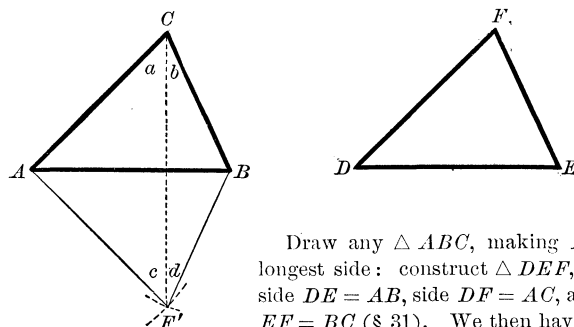
5. Then, $\angle A = \angle B$.

[In equal figures, the homologous parts are equal.] (§ 48)

51. It follows from the preceding that
An equilateral triangle is also equiangular.

PROP. V. THEOREM

52. *Two triangles are equal when the three sides of one are equal respectively to the three sides of the other.*



Draw any $\triangle ABC$, making AB the longest side: construct $\triangle DEF$, having side $DE = AB$, side $DF = AC$, and side $EF = BC$ (§ 31). We then have:

Given in $\triangle ABC$ and DEF , $AB = DE$, $BC = EF$, and $AC = DF$.

To Prove $\triangle ABC = \triangle DEF$.

Proof. 1. Let AB be longest side of $\triangle ABC$; and place $\triangle DEF$ in position ABF' , side DE coinciding with its equal AB , and vertex F falling at F' , on the opposite side of AB from C .

2. Draw line CF' , and represent $\angle ACF'$, BCF' , $AF'C$, and $BF'C$ by a , b , c , and d , respectively.

3. Since $AC = AF'$, $\angle a = \angle c$. (1)

[In an isosceles $\triangle$, the $\angle$ opposite the equal sides are equal.] (§ 51)

4. Since $BC = BF'$, $\angle b = \angle d$. (2)

5. Adding (1) and (2),

$$\angle a + \angle b = \angle c + \angle d; \text{ or } \angle ACB = \angle AF'B.$$

6. Since sides AC and BC and $\angle ACB$, of $\triangle ABC$, are equal, respectively, to sides AF' and BF' and $\angle AF'B$, of $\triangle ABF'$,

$$\triangle ABC = \triangle ABF'.$$

[Two $\triangle$ are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 46)

7. That is, $\triangle ABC = \triangle DEF$.

53. Note. We may now see the reason for the construction given in § 28. We know that the triangle whose vertices are the points B , G , and H has its sides respectively equal to those of the triangle whose vertices are the points E , K , and L .

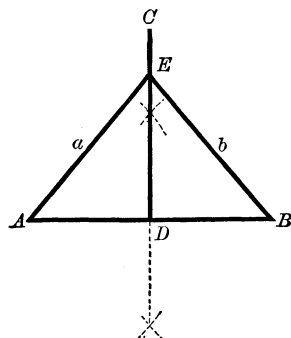
Then, the triangles are equal by § 52, and the homologous angles B and E are equal.

PROP. VI. THEOREM

54. *If a perpendicular be erected at the middle point of a straight line,*

I. *Any point in the perpendicular is equally distant from the extremities of the line.*

II. *Any point without the perpendicular is unequally distant from the extremities of the line.*



At the middle point D of any line AB erect a $\perp DC$. From E , any point in DC , draw lines to A and B . We now have :

I. **Given** line $CD \perp$ to line AB at its middle point D , E any point in CD , and lines AE (a) and BE (b).

To Prove $a = b$.

Proof. 1. In $\triangle ADE$ and BDE , $DE = DE$.

2. By hyp., $AD = DB$.

3. Also, $\angle ADE = \angle BDE$.

[All rt. $\angle$ s are equal.]

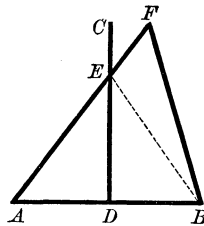
(§ 16)

4. Then, $\triangle ADE = \triangle BDE$.

[Two $\triangle$ are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 46)

5. Then, $a = b$.

[In equal figures, the homologous parts are equal.] (§ 48)



Draw line AB . At D , the middle point of AB , draw $DC \perp AB$. From F , any point without CD , draw lines FA and FB . We then have :

II. **Given** line $CD \perp$ to line AB at its middle point D , F any point without CD , and lines AF and BF .

To Prove AF and BF unequal.

Proof. 1. Let AF intersect CD at E , and draw line BE .

2. We have $BE + EF > BF$.

[A str. line is the shortest line between two points.] (Ax. 7)

3. But, $BE = AE$.

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 54, I)

4. Substituting for BE its equal AE ,

$$AE + EF > BF, \text{ or } AF > BF.$$

55. It follows from Prop. VI that *every point which is equally distant from the extremities of a straight line, lies in the perpendicular erected at the middle point of the line.*

56. A straight line is determined by any two of its points (§ 35); whence,

Two points, each equally distant from the extremities of a straight line, determine a perpendicular to that line at its middle point.

57. From the equal triangles ADE and BDE , in the figure of § 54, I,

$$\angle AED = \angle BED;$$

since homologous parts of equal figures are equal.

Then, if lines be drawn to the extremities of a straight line from any point in the perpendicular erected at its middle point, they make equal angles with the perpendicular.

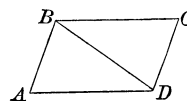
58. We may now see the reasons for the constructions given in §§ 24, 25, 26, and 27.

In § 24, by construction, points C and F are each equally distant from points D and E ; and CF is perpendicular to AB by § 56.

In § 25, points C and F are each equally distant from points D and E ; and in § 26, points C and D are each equally distant from points A and B .

In § 27, points O and E are each equally distant from points C and D ; then, OE is perpendicular to straight line CD at its middle point (§ 56), and $\angle AOE = \angle BOE$ by § 57.

Ex. 10. If in triangles ABD and BCD , sides AB , BD , and AD are equal, respectively, to sides CD , BD , and BC , what angles of the triangles are equal?



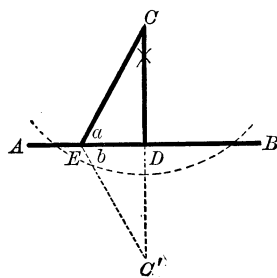
Ex. 11. Two lines of unequal length bisect each other at right angles. Show that any point in either line is equidistant from the extremities of the other. (§ 54.)

Ex. 12. If lines be drawn from the extremities of a straight line to any point in the perpendicular erected at its middle point, they make equal angles with the line. (Figure of § 54, I. Use § 48.)

PROP. VII. THEOREM

59. *From a given point without a straight line, but one perpendicular can be drawn to the line.*

(It follows from § 25 that, from a given point without a straight line, a perpendicular can be drawn to the line.)



From point C without line AB , draw $CD \perp AB$ (§ 25). We then have :

Given point C without line AB , and line $CD \perp AB$.

To Prove CD the only $\perp$ which can be drawn from C to AB .

Proof. 1. If possible, let CE be another $\perp$ from C to AB .

2. Extend CD to C' , making $C'D = CD$, and draw line EC' ; represent $\angle CED$ by a , and $\angle C'ED$ by b .

3. Since ED is $\perp CC'$ at its middle point D , $\angle a = \angle b$.

[If lines be drawn to the extremities of a str. line from any point in the $\perp$ erected at its middle point, they make equal $\angle$ s with the $\perp$.] (§ 57)

4. By hyp., $\angle a$ is a rt. $\angle$.

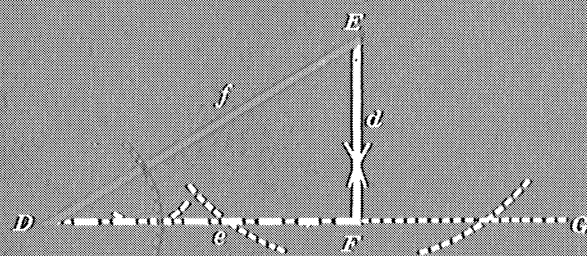
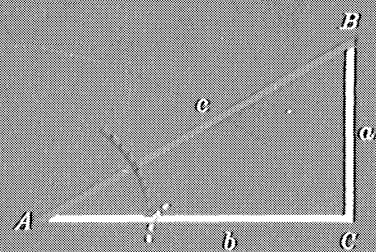
5. Then, $\angle b$ is a rt. $\angle$, and $\angle a + \angle b =$ two rt. $\angle$ s.

6. Then line CEC' is a str. line.

[If the sum of two adj. $\angle$ s is equal to two rt. $\angle$ s, their ext. sides lie in the same str. line.] (§ 20)

7. This is impossible, for, by cons., CDC' is a str. line.

[But one str. line can be drawn between two points.] (Ax 6.)



PROP. VIII.

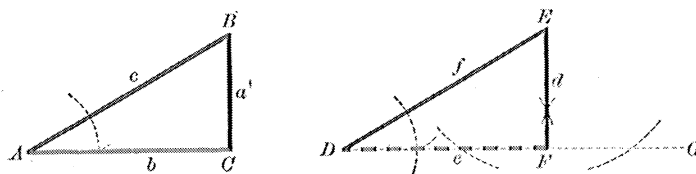
PLATE III

8. Hence, CE cannot be $\perp AB$, and CD is the only $\perp$ that can be drawn.

Ex. 13. If AB, AC , and DB, DC are the equal sides of two unequal isosceles triangles ABC, DBC , having common side BC , the line joining A and D bisects BC .

PROP. VIII. THEOREM

60. *Two right triangles are equal when the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other.*



Draw any $\triangle ABC$, right angled at C ; draw line $DE = AB$; construct $\angle D = \angle A$; draw line $EF \perp DG$. We now have:

Given, in rt. $\triangle ABC$ and DEF ,

hypotenuse $AB (c) = \text{hypotenuse } DE (f)$,

and $\angle A = \angle D$.

To Prove $\triangle ABC = \triangle DEF$.

Proof. 1. Denote BC by a , AC by b , EF by d , and DF by e .

2. Superpose $\triangle ABC$ upon $\triangle DEF$ in such a way that hypotenuse c shall coincide with its equal f ; vertex A falling on vertex D , and vertex B on vertex E .

3. Since $\angle A = \angle D$, side b will fall on side e .

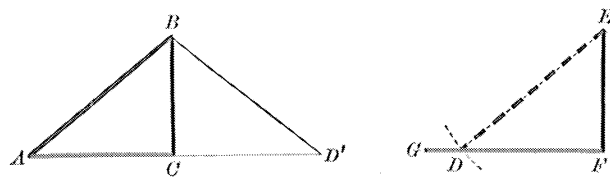
4. Then, side a will fall on side d .

[From a given point without a str. line, but one $\perp$ can be drawn to the line.] (§ 59)

5. Therefore, the $\triangle$ coincide throughout, and are equal.

PROP. IX. THEOREM

61. *Two right triangles are equal when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.*



Draw any $\triangle ABC$, right angled at C ; draw line $EF = BC$; construct rt. $\angle EFG$; with E as centre and AB as radius describe an arc intersecting FG at D ; draw line DE . We now have :

Given, in rt. $\triangle ABC$ and DEF ,

hypotenuse $AB =$ hypotenuse DE , and $BC = EF$.

To Prove $\triangle ABC = \triangle DEF$.

Proof. 1. Place $\triangle DEF$ in position BCD' ; vertex E falling at B , F at C , and D at D' , on opposite side of BC from A .

2. Since ACB and BCD' are rt. $\angle$ s, ACD' is a str. line.

[If the sum of two adj. $\angle$ s is equal to two rt. $\angle$ s, their ext. sides lie in the same str. line.] (§ 20)

3. Then, $\triangle ABD'$ is isosceles, and $\angle A = \angle D'$.

[In an isosceles $\triangle$, the $\angle$ opposite the equal sides are equal.] (§ 50)

4. Then, $\triangle ABC = \triangle BCD'$.

[Two rt. $\triangle$ s are equal when the hypotenuse and an adj. $\angle$ of one are equal respectively to the hypotenuse and an adj. $\angle$ of the other.] (§ 60)

5. That is, $\triangle ABC = \triangle DEF$.

PROP. X. THEOREM

62. *Any side of a triangle is greater than the difference of the other two sides.*

Draw $\triangle ABC$ with side $BC >$ side AC . We now have :

Given AB , any side of $\triangle ABC$, and $BC > AC$.

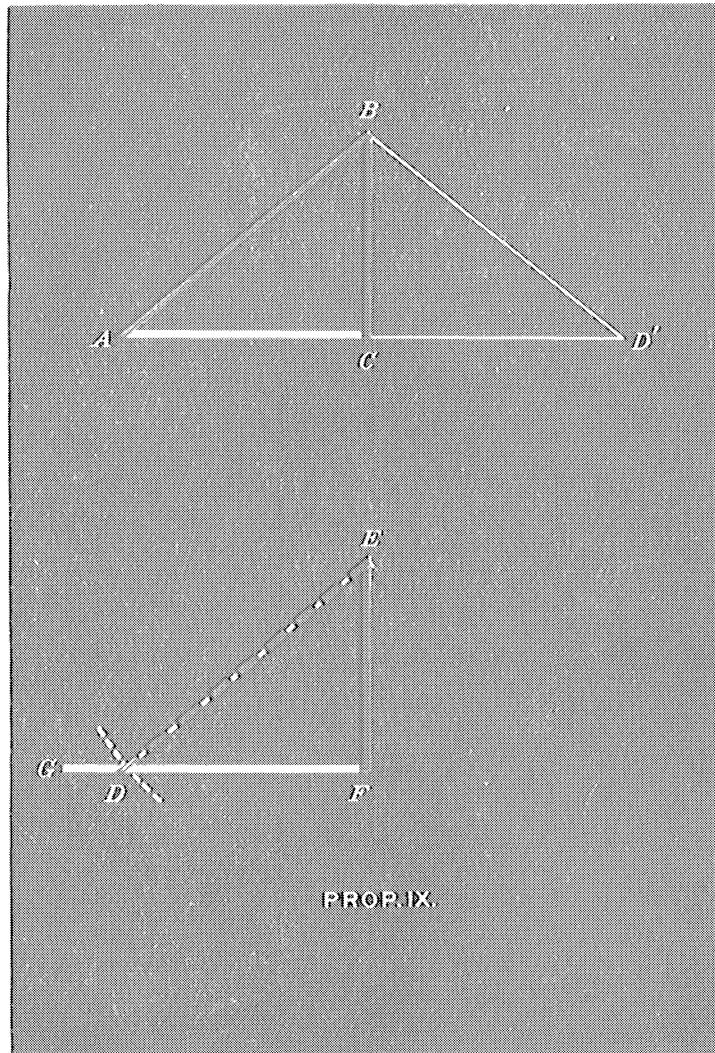


PLATE IV

To Prove $AB > BC - AC$.

Proof. 1. We have $AB + AC > BC$.

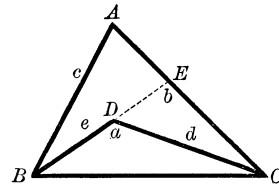
[A str. line is the shortest line between two points.] (Ax. 7)

2. Subtracting AC from both members of the inequality,

$$AB > BC - AC.$$

PROP. XI. THEOREM

63. *The sum of any two sides of a triangle is greater than the sum of the lines drawn from any point within the triangle to the extremities of the remaining side.*



Draw $\triangle ABC$. From D any point within the $\triangle$ draw lines DC and DB . Represent AC , AB , DC , and DB by b , c , d , and e , respectively. We then have:

Given d and e lines drawn from any point D within $\triangle ABC$ to the extremities of side BC .

To Prove $b + c > d + e$.

Proof. 1. Extend BD to meet AC at E .

2. We have $c + AE > e + DE$. (1)

[A str. line is the shortest line between two points.] (Ax. 7)

3. For same reason, $CE + DE > d$. (2)

4. Adding inequalities (1) and (2), and observing that $AE + CE = b$, we have

$$b + c + DE > d + e + DE.$$

5. Subtracting DE from both members of the inequality,

$$b + c > d + e.$$

Ex. 14. Given one side of a triangle, and the perpendicular drawn to it from the vertex of the opposite angle, to construct the triangle. Can more than one such triangle be drawn?

Ex. 15. There are six elements in every triangle, — three sides and three angles. How many of these elements and what elements are needed that only one definite triangle can be constructed?

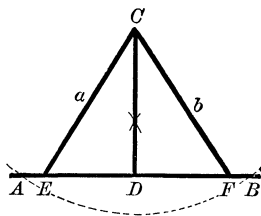
Ex. 16. Triangles are to be formed by choosing any three lines from six given lines whose lengths are 2, 3, 4, 5, 6, 7 inches, respectively. How many triangles can you form and what kind of triangles are they?

PROP. XII. THEOREM

64. *If oblique lines be drawn from a point to a straight line,*

I. *Two oblique lines cutting off equal distances from the foot of the perpendicular from the point to the line are equal.*

II. *Of two oblique lines cutting off unequal distances from the foot of the perpendicular from the point to the line, the more remote is the greater.*



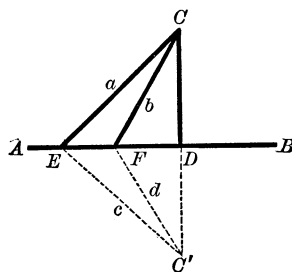
I. From any point C without line AB , draw $CD \perp AB$ (§ 25); take line $ED = DF$ and draw lines CE (a) and CF (b). We then have:

I. **Given** $CD \perp$ from point C to line AB ; and lines a and b cutting off equal distances from the foot of CD .

To Prove $a = b$.

Proof. Since CD is $\perp EF$ at its middle point D , $a = b$.

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 54)



II. Construct the figure in accordance with the statement. We then have :

II. **Given** CD the $\perp$ from point C to line AB ; and CE (a) and CF (b) oblique lines from C to AB , cutting off unequal distances from the foot of CD ; a being the more remote.

To Prove $a > b$.

Proof. 1. Produce CD to C' , making $C'D = CD$, and draw lines $C'E$ (c) and $C'F$ (d).

2. Since by cons., AD is $\perp$ CC' at its middle point D ,

$$a = c, \text{ and } b = d.$$

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 54)

3. But $a + c > b + d$.

[The sum of any two sides of a Δ is $>$ the sum of the lines drawn from any point within the Δ to the extremities of the remaining side.] (§ 63)

4. Substituting for c and d their equals a and b , respectively,

$$2a > 2b.$$

5. Dividing by 2, $a > b$.

Note. The theorem holds equally if CE is on the opposite side of CD from CF .

Ex. 17. Two sides of a triangle are 8 inches and 10 inches, respectively. Between what limits must the third side lie?

PROP. XIII. THEOREM

65. *If oblique lines be drawn from a point to a straight line,*

I. *Two equal oblique lines cut off equal distances from the foot of the perpendicular from the point to the line.*

II. *Of two unequal oblique lines, the greater cuts off the greater distance from the foot of the perpendicular from the point to the line.*

I. Draw line AB and take C any point without AB . Draw line $CD \perp AB$. With C as centre and radius greater than CD describe arc intersecting AD at E , and BD at F ; draw lines CE (a) and CF (b). We then have :

I. **Given** $CD \perp$ from point C to line AB , and a and b equal oblique lines from C to AB .

To Prove $ED = DF$.

Proof. 1. In rt. $\triangle CDE$ and CDF , $CD = CD$.

2. By hyp., $a = b$.

3. Then, $\triangle CDE = \triangle CDF$.

[Two rt. $\triangle$ are equal when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.] (§ 61)

4. Then, $DE = DF$.

[In equal figures, the homologous parts are equal.] (§ 48)

II. Draw line AB and take C any point without AB . Draw line $CD \perp AB$; draw lines CE (a) and CF (b) intersecting AD at E and F , respectively, making $a > b$. We then have :

II. **Given** CD the $\perp$ from point C to line AB ; and a and b unequal oblique lines from C to AB , a being $> b$.

To Prove $DE > DF$.

Proof. 1. We know that DE is either $<$, $=$, or $> DF$.

2. If we suppose $DE < DF$, a would be $< b$.

[If oblique lines be drawn from a point to a str. line, of two oblique lines cutting off unequal distances from the foot of the $\perp$ from the point to the line, the more remote is the greater.] (§ 64)

3. But this is contrary to the hypothesis that a is $> b$; then DE cannot be $< DF$.

4. If we suppose $DE = DF$, a would be $= b$.

[If oblique lines be drawn from a point to a str. line, two oblique lines cutting off equal distances from the foot of the $\perp$ from the point to the line are equal.] (§ 64)

5. This is contrary to the hypothesis that a is $> b$; then DE cannot be $= DF$.

6. If DE cannot be $< DF$, nor $= DF$, we must have

$$DE > DF.$$

66. Note I. The method of proof used in Prop. XIII is called the Indirect Method, or the *Reductio ad absurdum*.

We prove a proposition by making every possible supposition in regard to it; and showing that, in every case except the one we wish to prove, the supposition leads to something contrary to the hypothesis.

67. Note II. We may state Prop. XII, I, as follows:

Hypothesis. If two oblique lines be drawn from a point to a straight line, cutting off equal distances from the foot of the perpendicular from the point to the line,

Conclusion. They are equal.

Again, we may state Prop. XIII, I,

Hypothesis. If two equal oblique lines be drawn from a point to a straight line,

Conclusion. They cut off equal distances from the foot of the perpendicular from the point to the line.

We call one proposition the **converse** of another when the hypothesis and conclusion of the first are, respectively, the conclusion and hypothesis of the second.

It follows from the above that Prop. XIII, I, is the *converse* of Prop. XII, I.

Prop. XIII, II, is the converse of Prop. XII, II; also § 20 is the converse of § 17.

PARALLEL LINES

68. Def. Two straight lines are said to be *parallel* ($\parallel$) when they lie in the same plane, and cannot meet how-
 $A \text{-----} B$
 ever far they may be produced; as AB and CD . $C \text{-----} D$

69. Ax. We assume that but one straight line can be drawn through a given point parallel to a given straight line.

PROP. XIV. THEOREM

70. Two perpendiculars to the same straight line are parallel.

Draw a line EG ; at points A and C , respectively, on EG , draw lines AB and $CD \perp EG$. We now have :

Given lines AB and $CD \perp$ to line AC .

To Prove $AB \parallel CD$.

Proof. 1. If AB and CD are not $\parallel$, they will meet in some point if sufficiently produced (§ 68).

2. We should then have two $\perp$ s from this point to AC , which is impossible.

[From a given point without a str. line, but one $\perp$ can be drawn to the line.] (§ 59)

3. Therefore, AB and CD cannot meet, and are $\parallel$.

PROP. XV. THEOREM

71. Two straight lines parallel to the same straight line are parallel to each other.

Draw line EF . Draw lines AB and $CD \parallel$ to EF (§ 70). We then have :

Given lines AB and $CD \parallel$ to line EF .

To Prove $AB \parallel CD$.

Proof. 1. If AB and CD are not $\parallel$, they will meet in some point if sufficiently produced. (§ 68)

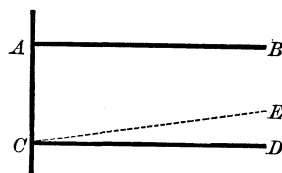
2. We should then have two lines drawn through this point $\parallel$ to EF , which is impossible.

[But one str. line can be drawn through a given point $\parallel$ to a given str. line.] (§ 69)

3. Therefore, AB and CD cannot meet, and are $\parallel$.

PROP. XVI. THEOREM

72. *A straight line perpendicular to one of two parallels is perpendicular to the other.*



Draw line $AB \parallel CD$ (§ 70); draw line $AC \perp AB$. We then have :

Given lines AB and $CD \parallel$, and line $AC \perp AB$.

To Prove $AC \perp CD$.

Proof. 1. If CD is not $\perp AC$, let line CE be $\perp AC$.

2. Then since AB and CE are $\perp AC$, $CE \parallel AB$.

[Two $\perp$ s to the same str. line are $\parallel$.]

(§ 70)

3. But by hyp., $CD \parallel AB$.

4. Then, CE must coincide with CD .

[But one str. line can be drawn through a given point $\parallel$ to a given str. line.]

(§ 69)

5. Now by cons., $AC \perp CE$.

6. Then since CE coincides with CD , we have $AC \perp CD$.

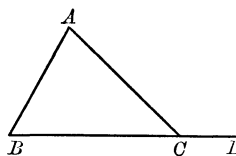
DEFINITIONS

73. An *exterior angle* of a triangle is the angle at any vertex formed by any side of the triangle and the adjacent side produced; as ACD .

If any side of a triangle be taken and called the *base*, we define the corresponding *altitude* as the perpendicular drawn from the opposite vertex to the base, produced if necessary.

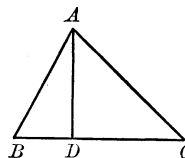
In general, any side may be taken as the base; but in an isosceles triangle, unless otherwise specified, the side which is not one of the equal sides is taken as the base.

When any side has been taken as the base, we call the



opposite angle the *vertical angle*, and its vertex the *vertex of the triangle*.

Thus, in triangle ABC , BC is the base, AD the altitude, and BAC the vertical angle.



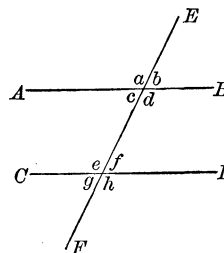
74. If two straight lines, AB and CD , are cut by a line EF , called a *transversal*, the angles are named as follows:

c , d , e , and f are called *interior* angles, and a , b , g , and h *exterior* angles.

c and f , or d and e , are called *alternate-interior* angles.

a and h , or b and g , are called *alternate-exterior* angles.

a and e , b and f , c and g , or d and h , are called *corresponding* angles.



PROP. XVII. THEOREM

75. If two parallels are cut by a transversal, the *alternate-interior* angles are equal.

Draw line $AB \parallel CD$ (§ 70). Draw any line EF (not $\perp AB$) intersecting AB and CD at G and H , respectively. We then have:

Given $\parallel$ s AB and CD cut by transversal EF at points G and H , respectively, forming alt. int. $\angle$ s AGH (a) and GHD (c), also BGH (b) and CHG (d).

To Prove $\angle a = \angle c$, and $\angle b = \angle d$.

Proof. 1. Through K , the middle point of GH , draw a line $\perp AB$, meeting AB at L , and CD at M .

2. Then, $LM \perp CD$.

[A str. line $\perp$ to one of two $\parallel$ s is $\perp$ to the other.] (§ 72)

3. In rt. $\triangle GKL$ and HKM , by cons.,

hypotenuse $GK =$ hypotenuse HK .

4. Also, $\angle GKL = \angle HKM$.

[If two str. lines intersect, the vertical $\angle$ s are equal.] (§ 44)

5. Then $\triangle GKL = \triangle HKM$.

[Two rt. $\triangle$ are equal when the hypotenuse and an adj. $\angle$ of one are equal respectively to the hypotenuse and an adj. $\angle$ of the other.] (§ 60)

6. Then, $\angle a = \angle c$.

[In equal figures, the homologous parts are equal.] (§ 48)

7. Again, $\angle a$ is the supplement of $\angle b$, and $\angle c$ of $\angle d$.

[If two adj. $\angle$ s have their ext. sides in the same str. line, their sum is equal to two rt. $\angle$ s.] (§ 17)

8. Then, since $\angle a = \angle c$, we have $\angle b = \angle d$.

[The supplements of equal $\angle$ s are equal.] (§ 41)

76. We have $\angle a = \angle BGE$. (Fig. of Prop. XVII.)

[If two str. lines intersect, the vertical $\angle$ s are equal.] (§ 44)

Then, since $\angle a = \angle c$, we have $\angle BGE = \angle c$.

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

In like manner,

$$\angle AGE = \angle d, \angle CHF = \angle a, \text{ and } \angle DHF = \angle b.$$

That is, *if two parallels are cut by a transversal, the corresponding angles are equal.*

77. We have $\angle a + \angle b = \text{two rt. } \angle$ s. (Fig. of Prop. XVII.)

[If two adj. $\angle$ s have their ext. sides in the same str. line, their sum is equal to two rt. $\angle$ s.] (§ 17)

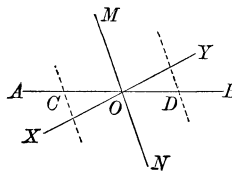
Putting for $\angle b$ its equal $\angle d$, we have

$$\angle a + \angle d = \text{two rt. } \angle$$

In like manner, $\angle b + \angle c = \text{two rt. } \angle$ s.

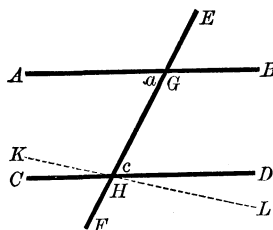
That is, *if two parallels are cut by a transversal, the sum of the interior angles on the same side of the transversal is equal to two right angles.*

Ex. 18. A line AB intersects line XY at O , and through O any line MN is drawn; if through points C and D on AB equidistant from O , parallels to MN be drawn, the triangles formed are equal.



PROP. XVIII. THEOREM

78. (Converse of Prop. XVII.) *If two straight lines are cut by a transversal, and the alternate-interior angles are equal, the two lines are parallel.*



Draw line AB . Draw line EF cutting AB at G . Through any point H , in EF , draw line CD making $\angle GHD$ (c) equal $\angle AGH$ (a). We then have :

Given lines AB and CD cut by transversal EF at points G and H , respectively, and

$$\angle a = \angle c.$$

To Prove $AB \parallel CD$.

Proof. 1. If CD is not $\parallel AB$, draw line KL through $H \parallel AB$.
2. Then since $\parallel AB$ and KL are cut by transversal EF ,

$$\angle a = \angle GHL.$$

[If two $\parallel$ s are cut by a transversal, the alt. int. $\angle$ s are equal.] (§ 75)

3. But by hyp., $\angle a = \angle c$.

4. Then, $\angle GHL = \angle c$.

[Things which are equal to the same thing are equal to each other.]
(Ax. 1)

5. But this is impossible unless KL coincides with CD .

6. Then, $CD \parallel AB$.

In like manner it may be proved that if AB and CD are cut by EF , and $\angle BGH = \angle CHG$, then $AB \parallel CD$.

79. (Converse of § 76.) *If two straight lines are cut by a transversal, and the corresponding angles are equal, the two lines are parallel.*

Suppose $\angle a = \angle CHF$. (Fig. of Prop. XVIII.)

Now, $\angle c = \angle CHF$.

[If two str. lines intersect, the vertical $\angle$ s are equal.] (§ 44)

Then, $\angle a = \angle c$.

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

Then, by § 78, $AB \parallel CD$.

In like manner we may prove that if

$$\angle BGE = \angle c, \text{ or } \angle AGE = \angle CHG, \text{ or } \angle BGH = \angle DHF,$$

then, $AB \parallel CD$.

80. (Converse of § 77.) *If two straight lines are cut by a transversal, and the sum of the interior angles on the same side of the transversal is equal to two right angles, the two lines are parallel.*

Suppose

$$\angle BGH + \angle c = \text{two rt. } \angle\text{s. (Fig. of Prop. XVIII.)}$$

Then, both $\angle a$ and $\angle c$ are supplements of $\angle BGH$,

and $\angle a = \angle c$.

[The supplements of equal $\angle$ s are equal.] (§ 41)

Then, by § 78, $AB \parallel CD$.

In like manner it may be proved that if

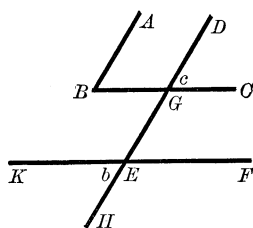
$$\angle a + \angle CHG = \text{two rt. } \angle\text{s, then, } AB \parallel CD.$$

Ex. 19. If two parallels are cut by a transversal, the alternate exterior angles are equal.

Ex. 20. If two straight lines are cut by a transversal, and the alternate-exterior angles are equal, the two lines are parallel.

PROP. XIX. THEOREM

81. *Two angles whose sides are parallel, each to each, are equal if both pairs of parallel sides extend in the same direction, or in opposite directions, from their vertices.*



Draw lines AB and $BC \parallel$ to lines DH and KF , respectively, intersecting at E . We then have :

Given $\angle DEF$ (a), with its sides $\parallel$ to and extending in the *same* direction as those of $\angle B$, and $\angle HEK$ (b), with its sides $\parallel$ to and extending in the *opposite* direction to those of $\angle B$.

To Prove $\angle B = \angle a$, and $\angle B = \angle b$.

Proof. 1. Let BC and DH intersect at G ; denote $\angle DGC$ by c .

2. Since $\parallel$ s AB and DE are cut by BC ,
 $\angle B = \angle c$.

[If two $\parallel$ s are cut by a transversal, the corresp. $\angle$ s are equal.] (§ 76)

3. For same reason, since $\parallel$ s BC and EF are cut by DE ,

$$\angle c = \angle a.$$

4. Then, $\angle B = \angle a$.

[Things which are equal to the same thing are equal to each other.]

(Ax. 1)

5. Again, $\angle a = \angle b$.

[If two str. lines intersect, the vertical $\angle$ s are equal.] (§ 44)

6. Then, $\angle B = \angle b$.

[Things which are equal to the same thing are equal to each other.]

(Ax. 1)

Note. The sides extend in the *same* direction if they are on the *same* side of a straight line joining the vertices, and in *opposite* directions if they are on *opposite* sides of this line.

82. We have $\angle a$ the supplement of $\angle DEK$. (Fig. of Prop. XIX.)

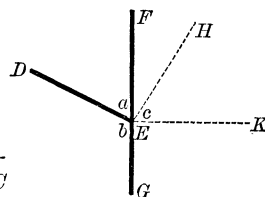
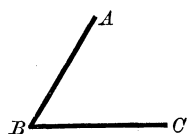
[If two adj. $\angle$ s have their ext. sides in the same str. line, their sum is equal to two rt. $\angle$ s.] (§ 17)

Then its equal, $\angle B$, is the supplement of $\angle DEK$.

That is, *two angles whose sides are parallel, each to each, are supplementary if one pair of parallel sides extend in the same direction, and the other pair in opposite directions, from their vertices.*

PROP. XX. THEOREM

83. *Two angles whose sides are perpendicular, each to each, are either equal or supplementary.*



Draw line FG ; also, line DE meeting FG at E . Draw lines AB and BC $\perp$ to lines DE and FG , respectively.

We then have :

Given $\angle DEF$ (a), $\angle DEG$ (b); $DE \perp AB$, and $FG \perp BC$.

To Prove $\angle B$ equal to $\angle a$, and supplementary to $\angle b$.

Proof. 1. Draw line $EH \perp DE$, and line $EK \perp EF$; denote $\angle HEK$ by c .

2. Since EH and AB are $\perp$ DE , $EH \parallel AB$.

[Two $\perp$ s to the same str. line are $\parallel$.]

(§ 70)

3. Since EK and BC are $\perp$ FG , $EK \parallel BC$.

4. Then, $\angle c = \angle B$.

[Two $\angle$ s whose sides are $\parallel$, each to each, are equal if both pairs of $\parallel$ sides extend in the same direction from their vertices.] (§ 81)

5. Since, by cons., $\angle DEH$ and FEK are rt. $\angle$ s, each of the $\angle$ s a and c is the complement of $\angle FEH$.

6. Then, $\angle a = \angle c$.

[The complements of equal $\angle$ s are equal.]

(§ 41)

7. Then, $\angle B = \angle a$.

[Things which are equal to the same thing are equal to each other.]

(Ax. 1)

8. Again, $\angle a$ is the supplement of $\angle b$.

[If two adj. $\angle$ s have their ext. sides in the same str. line, they are supplementary.]

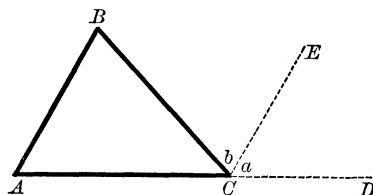
(§ 17)

9. Then, its equal, $\angle B$, is the supplement of $\angle b$.

Note. The angles are equal if they are both acute or both obtuse; and supplementary if one is acute and the other obtuse.

PROP. XXI. THEOREM

84. *The sum of the angles of any triangle is equal to two right angles.*



Given $\triangle ABC$.

To Prove $\angle A + \angle B + \angle C = \text{two rt. } \angle$ s.

Proof. 1. Extend AC to D , and draw line $CE \parallel AB$; represent $\angle ECD$ by a , and $\angle BCE$ by b .

2. We have $\angle a + \angle b + \angle ACB = \text{two rt. } \angle$ s. (1)

[The sum of all the $\angle$ s on the same side of a str. line at a given point is equal to two rt. $\angle$ s.]

(§ 18)

3. Since $\parallel$ s AB and CE are cut by AD ,

$$\angle a = \angle A.$$

[If two $\parallel$ s are cut by a transversal, the corresp. $\angle$ s are equal.]

(§ 76)

4. Since $\parallel$ s AB and CE are cut by BC ,

$$\angle b = \angle B.$$

[If two $\parallel$ s are cut by a transversal, the alt. int. $\angle$ s are equal.]

(§ 75)

5. Substituting in (1) $\angle A$ for $\angle a$ and $\angle B$ for $\angle b$,

$$\angle A + \angle B + \angle ACB = \text{two rt. } \angle.$$

85. We have, from § 84, $\angle BCD = \angle a + \angle b = \angle A + \angle B$.

Hence, *an exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

The following are immediate consequences of §§ 84, 85:

86. *An exterior angle of a triangle is greater than either of the opposite interior angles.*

87. *If two triangles have two angles of one equal, respectively, to two angles of the other, the third angle of the first is equal to the third angle of the second.*

88. *A triangle cannot have two right angles, nor two obtuse angles.*

89. *Two right triangles are equal when a leg and an acute angle of one are equal, respectively, to a leg and the homologous acute angle of the other.* For the remaining acute angles are equal by § 87, and the triangles are equal by § 46.

Ex. 21. Can a triangle be formed whose angles are 35° , 65° , and 55° , respectively? 85° , 50° , and 75° ? 50° , 75° , and 55° ? $45^\circ 11' 20''$, $61^\circ 52' 48''$, and $72^\circ 55' 52''$?

Ex. 22. The sum of one of the base angles of an isosceles triangle and the vertical angle equals 140° ; find the degrees in the exterior angle formed by producing one of the equal sides through the vertex.

PROP. XXII. THEOREM

90. (Converse of Prop. IV.) *If two angles of a triangle are equal, the sides opposite are equal.*

Draw $\triangle ABC$ with $\angle A = \angle B$. Represent BC by a and AC by b .

Given, in $\triangle ABC$, $\angle A = \angle B$.

To Prove $a = b$.

The proof is left to the pupil. Draw line $CD \perp AB$. The $\triangle ACD$ and BCD are equal by § 89.

91. From equal $\triangle ACD$ and BCD (Fig. of Prop. XXII),

$$AD = BD, \text{ and } \angle ACD = \angle BCD.$$

Hence, *the perpendicular from the vertex to the base of an isosceles triangle bisects the base and also the vertical angle.*

PROP. XXIII. THEOREM

92. *If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle lies opposite the greater side.*

Draw $\triangle ABC$, with $BC (a) > AC (b)$. We then have :

Given $\triangle ABC$ with side $a >$ side b .

To Prove $\angle BAC > \angle B$.

Proof. 1. On CB take $CD = b$, and draw line AD .

2. In isosceles $\triangle ACD$, $\angle CAD = \angle CDA$.

[In an isosceles $\triangle$ the $\angle$ opposite the equal sides are equal.] (§ 50)

3. Since CDA is an ext. $\angle$ of $\triangle ABD$,

$$\angle CDA > \angle B.$$

[An ext. $\angle$ of a $\triangle$ is $>$ either of the opposite int. $\angle$ s.] (§ 86)

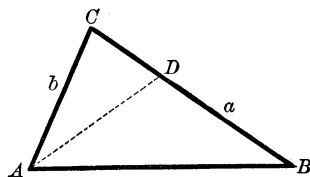
4. Then its equal, $\angle CAD$, is $> \angle B$.

5. Now, $\angle BAC$ is $> \angle CAD$.

6. Then, $\angle BAC$ must be $> \angle B$.

PROP. XXIV. THEOREM

93. (Converse of Prop. XXIII.) *If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side lies opposite the greater angle.*



Draw $\triangle ABC$ having $\angle BAC > \angle B$. We then have :

Given, in $\triangle ABC$, $\angle BAC > \angle B$.

To Prove $BC (a) > AC (b)$.

Proof. 1. Draw line AD making $\angle BAD = \angle B$, meeting BC at D .

2. In $\triangle ABD$, $AD = BD$.

[If two $\angle$ s of a $\triangle$ are equal, the sides opposite are equal.] (§ 90)

3. But $CD + AD > b$.

[A str. line is the shortest line between two points.] (Ax. 7)

4. Putting for AD its equal BD ,

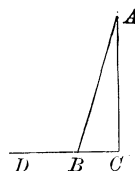
$$CD + BD > b, \text{ or } a > b.$$

The following are immediate consequences of § 93 :

94. *The hypotenuse of a right triangle is its greatest side.*

95. *The perpendicular is the shortest line which can be drawn from a point to a straight line.*

For, if AC is the $\perp$, and AB any other line, from A to CD , AB is the hypotenuse of rt. $\triangle ABC$; and hence is $> AC$ (§ 94).



Ex. 23. Two isosceles triangles are equal, if their vertical angles and their bases are respectively equal.

PROP. XXV. THEOREM

96. *Two parallel lines are everywhere equally distant.*

Note. By the *distance* of a point from a line, we mean the length of the perpendicular from the point to the line.

Draw lines AB and $CD \parallel$ (§ 70). At E and F , any two points in AB , draw lines EG and $FH \perp CD$. We then have :

Given $\parallel$ s AB and CD , and also EG (a), and FH (b) $\perp CD$.

To Prove $a = b$.

Proof. 1. Draw line FG .

2. We have $a \perp AB$.

[A str. line $\perp$ to one of two $\parallel$ s is $\perp$ to the other.] (§ 72)

3. Then, in rt. $\triangle EFG$ and FGH ,

$$FG = FG.$$

4. And since $\parallel$ s AB and CD are cut by FG ,

$$\angle EFG = \angle FGH.$$

[If two $\parallel$ s are cut by a transversal, the alt. int. $\angle$ s are equal.] (§ 75)

5. Then $\triangle EFG = \triangle FGH$.

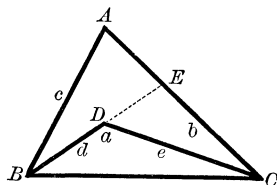
[Two rt. $\triangle$ s are equal when the hypotenuse and an adj. $\angle$ of one are equal respectively to the hypotenuse and an adj. $\angle$ of the other.] (§ 60)

6. Then $a = b$.

[In equal figures, the homologous parts are equal.] (§ 48)

PROP. XXVI. THEOREM

97. *If straight lines be drawn from a point within a triangle to the extremities of any side, the angle included by them is greater than the angle included by the other two sides.*



Draw any $\triangle ABC$; from any point D within the $\triangle$ draw lines DB and DC . We then have:

Given D , any point within $\triangle ABC$; and lines DB and DC forming $\angle BDC$ (a).

To Prove $\angle a > \angle A$.

Proof. 1. Extend BD to meet AC at E ; call $\angle DEC$ b .

2. Since a is an ext. $\angle$ of $\triangle CDE$,

$$\angle a > \angle b.$$

[An ext. $\angle$ of a $\triangle$ is $>$ either of the opposite int. $\angle$ s.] (§ 86)

3. Since b is an ext. $\angle$ of $\triangle ABE$,

$$\angle b > \angle A.$$

4. Since $\angle a > \angle b$, and $\angle b > \angle A$, we must have

$$\angle a > \angle A.$$

PROP. XXVII. THEOREM

98. *Any point in the bisector of an angle is equally distant from the sides of the angle.*

Construct any $\angle ABC$. Draw BD , the bisector of $\angle ABC$; from any point P in the bisector, draw lines PM (a) and PN (b) $\perp AB$ and BC , respectively. We then have:

Given P , any point in BD , the bisector of $\angle ABC$, and a and b lines from $P \perp AB$ and BC , respectively.

To Prove $a = b$.

Proof. 1. In rt. $\triangle BPM$ and BPN , $BP = BP$.

2. By hyp., $\angle PBM = \angle PBN$.

3. Then, $\triangle BPM = \triangle BPN$.

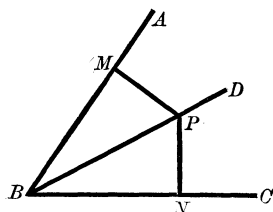
[Two rt. $\triangle$ are equal when the hypotenuse and an adj. $\angle$ of one are equal respectively to the hypotenuse and an adj. $\angle$ of the other.] (§ 60)

4. Then $PM = PN$.

[In equal figures, the homologous parts are equal.] (§ 48)

PROP. XXVIII. THEOREM

99. (Converse of Prop. XXVII.) *Every point which is within an angle and equally distant from its sides lies in the bisector of the angle.*



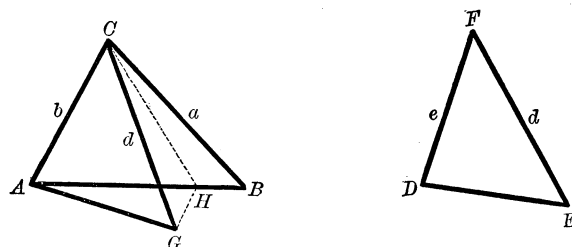
Given point P within $\angle ABC$, equally distant from sides AB and BC , and line BP .

To Prove $\angle PBM = \angle PBN$.

(Prove $\triangle BPM$ and BPN equal by § 61.)

PROP. XXIX. THEOREM

100. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.*



Draw $\triangle ABC$. Construct $\angle F < \angle ACB$. Draw lines FD (e) and FE (d) equal to CA (b) and CB (a), respectively. Draw line DE . We then have:

Given, in $\triangle ABC$ and DEF , $b = e$, $a = d$, and $\angle ACB > \angle F$.

To Prove $AB > DE$.

Proof. 1. Place $\triangle DEF$ in position ACG ; side e coinciding with its equal b , and vertex E falling at G .

2. Draw line CH bisecting $\angle GCB$, meeting AB at H ; also, line GH .

3. In $\triangle CGH$ and CBH , $CH = CH$.

4. By hyp., $d = a$, and $\angle GCH = \angle BCH$.

5. Then, $\triangle CGH = \triangle CBH$.

[Two $\triangle$ are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 46)

6. Then, $GH = BH$.

[In equal figures, the homologous parts are equal.] (§ 48)

7. Now, $AH + GH > AG$.

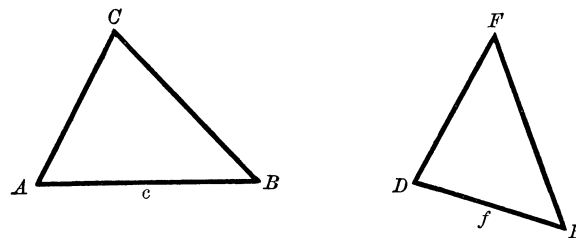
[A str. line is the shortest line between two points.] (Ax. 7)

8. Putting for GH its equal BH ,

$AH + BH > AG$, or $AB > DE$.

PROP. XXX. THEOREM

101. (Converse of Prop. XXIX.) *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second.*



Draw $\triangle ABC$. Draw $\triangle DEF$ making side $DF = AC$, side $EF = BC$, and side DE (f) $<$ AB (c). We then have :

Given, in $\triangle ABC$ and DEF , $AC = DF$, $BC = EF$, and $c > f$.

To Prove $\angle C > \angle F$.

Proof. 1. $\angle C$ must be $<$, $=$, or $> \angle F$.

2. If we suppose $\angle C = \angle F$, $\triangle ABC$ would equal $\triangle DEF$.

[Two $\triangle$ are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 46)

3. Then, c would equal f .

[In equal figures, the homologous parts are equal.] (§ 48)

4. Again, if we suppose $\angle C < \angle F$, c would be $< f$.

[If two $\triangle$ have two sides of one equal respectively to two sides of the other, but the included $\angle$ of the first $>$ the included $\angle$ of the second, the third side of the first is $>$ the third side of the second.] (§ 100)

5. Each of these conclusions is contrary to the hypothesis that c is $> f$.

6. If $\angle C$ cannot be $= \angle F$, nor $< \angle F$, we must have

$$\angle C > \angle F.$$

Ex. 24. ABC is an isosceles triangle. Lines AD and CE , intersecting at O , bisect the equal angles A and C . Prove $\triangle AOE = \triangle DOC$.

Ex. 25. By drawing a line through the vertex of a triangle parallel to the base, prove that the sum of the angles of a triangle is equal to two right angles.

Ex. 26. The line which joins the vertex of an isosceles triangle to the intersection of the bisectors of the exterior angles at the base is a perpendicular bisector of the base.

Ex. 27. If one of the equal sides of an isosceles triangle be extended through the vertex, a line through the vertex parallel to the base is a bisector of the exterior angle at the vertex.

Ex. 28. The bisectors of the base angles of an isosceles triangle meet in a point and form with the base an isosceles triangle.

Ex. 29. Through the middle point of one of the equal sides of an isosceles triangle a line parallel to the other equal side is drawn to meet the base and the bisector of the exterior angle at the vertex. Prove the two triangles formed equal.

Ex. 30. The bisectors of the base angles of an isosceles triangle form an angle of 110° . Find its vertical angle.

Ex. 31. Prove Prop. XXIX when point E falls *within* triangle ABC .

Ex. 32. The line which joins the vertex of an isosceles triangle to the intersection of the bisectors of the base angles, bisects the vertical angle.

Ex. 33. The line which bisects the vertical angle of an isosceles triangle bisects the base at right angles.

Ex. 34. Is it always possible to find a point in a given line which is equidistant from two given intersecting lines? Is there ever more than one such point?

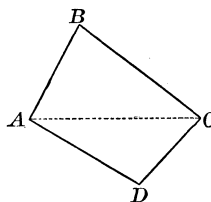
QUADRILATERALS

DEFINITIONS

102. A *quadrilateral* is a portion of a plane bounded by four straight lines; as $ABCD$.

The bounding lines are called the *sides* of the quadrilateral, and their points of intersection the *vertices*.

The *angles* of the quadrilateral are the angles formed by the adjacent sides, ABC , BCD , etc.



A *diagonal* is a straight line joining two opposite vertices ; as AC .

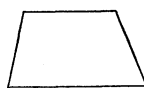
103. A *Trapezium* is a quadrilateral no two of whose sides are parallel.

A *Trapezoid* is a quadrilateral two, and only two, of whose sides are parallel.

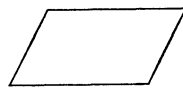
A *Parallelogram* is a quadrilateral whose opposite sides are parallel.



Trapezium



Trapezoid



Parallelogram

An *isosceles* trapezoid is a trapezoid whose non-parallel sides are equal.

The *bases* of a trapezoid are its parallel sides ; the *altitude* is the perpendicular distance between them.

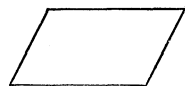
If either pair of parallel sides of a parallelogram be taken and called the *bases*, the *altitude* corresponding to these bases is the perpendicular distance between them.

A *Rhomboid* is a parallelogram whose angles are not right angles, and whose adjacent sides are unequal.

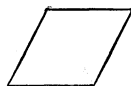
A *Rhombus* is a parallelogram whose angles are not right angles, and whose adjacent sides are equal.

A *Rectangle* is a parallelogram whose angles are right angles.

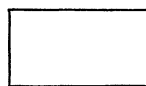
A *Square* is a rectangle whose sides are equal.



Rhomboid



Rhombus



Rectangle

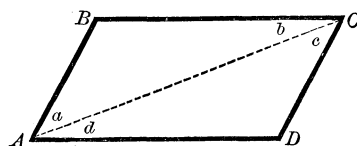


Square

Ex. 35. Bisect the exterior angles of a given triangle and join the vertices of the triangle thus formed to the opposite vertices of the given triangle. The lines thus drawn are the altitudes of the triangle formed by the bisectors.

PROP. XXXI. THEOREM

104. *In any parallelogram, the opposite sides are equal, and the opposite angles are equal.*



Draw the figure in accordance with the statement of the proposition. We then have :

Given $\square ABCD$.

To Prove

$AB = CD$, $BC = AD$, $\angle B = \angle D$, and $\angle BAD = \angle BCD$.

Proof. 1. Draw diagonal AC ; denote $\angle BAC$, BCA , ACD , and CAD by a , b , c , and d , respectively.

2. In $\triangle ABC$ and ACD , $AC = AC$.

3. Since $\parallel AB$ and CD are cut by AC ,

$$\angle a = \angle c. \quad (1)$$

[If two $\parallel$ s are cut by a transversal, the alt. int. $\angle$ s are equal.] (§ 75)

4. Since $\parallel BC$ and AD are cut by AC ,

$$\angle b = \angle d. \quad (2)$$

5. Then, $\triangle ABC = \triangle ACD$.

[Two $\triangle$ s are equal when a side and two adj. $\angle$ s of one are equal respectively to a side and two adj. $\angle$ s of the other.] (§ 49)

6. Then, $AB = CD$, $BC = AD$, and $\angle B = \angle D$.

[In equal figures, the homologous parts are equal.] (§ 48)
(Why are AB and CD , and BC and AD , homologous lines?)

7. Adding (1) and (2), we have

$$\angle a + \angle b = \angle c + \angle d, \text{ or } \angle BAD = \angle BCD.$$

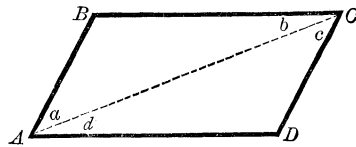
The following are consequences of § 104:

105. 1. *Parallel lines included between parallel lines are equal.*

2. *A diagonal of a parallelogram divides it into two equal triangles.*

PROP. XXXII. THEOREM

106. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Draw quadrilateral $ABCD$ having side $BC = AD$, and side $AB = DC$. We then have :

Given, in quadrilateral $ABCD$, $AB = CD$ and $BC = AD$.

To Prove $ABCD$ a $\square$.

Proof. 1. Draw diagonal AC ; represent $\angle BAC$, BCA , ACD , and CAD by a , b , c , and d , respectively.

2. In $\triangle ABC$ and ACD , $AC = AC$.

3. By hyp., $AB = CD$ and $BC = AD$.

4. Then, $\triangle ABC = \triangle ACD$.

[Two $\triangle$ are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 53)

5. Then, $\angle a = \angle c$, and $\angle b = \angle d$.

[In equal figures, the homologous parts are equal.] (§ 48)

6. Since $\angle a = \angle c$, $AB \parallel CD$.

[If two str. lines are cut by a transversal, and the alt. int. $\angle$ s are equal, the two lines are $\parallel$.] (§ 78)

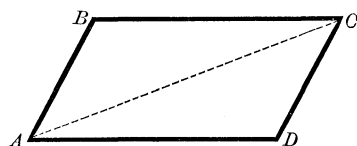
7. Since $\angle b = \angle d$, $BC \parallel AD$.

8. Then, by def., $ABCD$ is a $\square$.

Ex. 36. The perpendiculars from two opposite vertices of a parallelogram to a diagonal are equal.

PROP. XXXIII. THEOREM

107. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Draw line $BC =$ and $\parallel$ to AD , and lines AB and DC . We then have:

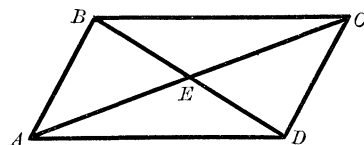
Given, in quadrilateral $ABCD$, BC equal and $\parallel$ to AD .

To Prove $ABCD$ a $\square$.

(Prove $\triangle ABC$ and ACD equal by § 46; then, $AB = CD$. Compare § 106.)

PROP. XXXIV. THEOREM

108. *The diagonals of a parallelogram bisect each other.*



Draw $\square ABCD$ as in § 107; and lines AC and BD . We then have:

Given diagonals AC and BD of $\square ABCD$ intersecting at E .

To Prove $AE = EC$ and $BE = ED$.

(Prove $\triangle AED = \triangle BEC$, by § 49.)

Note. The point E is called the *centre* of the parallelogram.

PROP. XXXV. THEOREM

109. (Converse of Prop. XXXIII.) *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

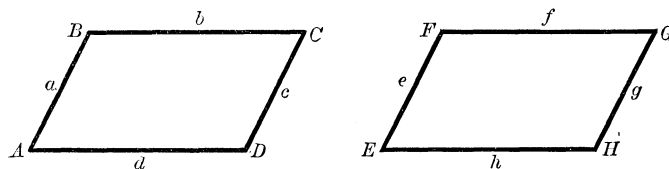
Given AC and BD , the diagonals of quadrilateral $ABCD$, bisecting each other at E . (Fig. of Prop. XXXIV.)

To Prove $ABCD$ a $\square$.

(Prove $\triangle AED = \triangle BEC$, by § 46; then $AD = BC$; in like manner, $AB = CD$; then use § 106.)

PROP. XXXVI. THEOREM

110. *Two parallelograms are equal when two adjacent sides and the included angle of one are equal respectively to two adjacent sides and the included angle of the other.*



Draw $\square ABCD$ and $EFGH$ in accordance with the statement of the proposition. Let a, b, c, d , represent sides AB, BC, CD , and DA , respectively; and e, f, g, h , sides EF, FG, GH , and HE , respectively. We then have:

Given, in $\square ABCD$ and $EFGH$,

$$a = e, d = h, \text{ and } \angle A = \angle E.$$

To Prove $\square ABCD = \square EFGH$.

Proof. 1. Superpose $\square ABCD$ upon $\square EFGH$ in such a way that $\angle A$ shall coincide with its equal $\angle E$; side a falling on side e , and side d on side h .

2. Since $a = e$, point B will fall on point F .

3. Since $d = h$, point D will fall on point H .

4. Since $b \parallel d$, and $f \parallel h$, side b will fall on side f , and point C will fall somewhere on f .

[But one str. line can be drawn through a given point $\parallel$ to a given str. line.] (§ 69)

5. Since $c \parallel a$, and $g \parallel e$, side c will fall on side g , and point C will fall somewhere on g .

6. Since C falls on both f and g , it must fall at their intersection, G .

7. Then, the $\square$ coincide throughout, and are equal.

The following is an immediate consequence of § 110:

111. *Two rectangles are equal if the base and altitude of one are equal respectively to the base and altitude of the other.*

PROP. XXXVII. THEOREM

112. *The diagonals of a rectangle are equal.*

Draw rectangle $ABCD$; draw lines AC and BD . We then have:

Given AC and BD the diagonals of rect. $ABCD$.

To Prove $AC = BD$.

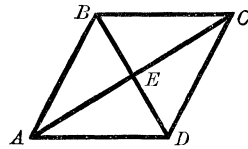
(Prove $\text{rt. } \triangle ABD = \text{rt. } \triangle ACD$, by § 46.)

The following is an immediate consequence of § 112:

113. *The diagonals of a square are equal.*

PROP. XXXVIII. THEOREM

114. *The diagonals of a rhombus bisect each other at right angles.*



Given AC and BD the diagonals of rhombus $ABCD$.

To Prove that AC and BD bisect each other at $\text{rt. } \angle$.
(Compare § 56.)

Ex. 37. The lines which join the middle points of the opposite sides of a parallelogram bisect each other.

Ex. 38. Are the diagonals of a parallelogram ever equal?

Ex. 39. When do the diagonals of a parallelogram bisect the opposite angles?

Ex. 40. If perpendiculars be drawn from the extremities of the upper base of an isosceles trapezoid to the lower base, they are equal and cut off equal segments from the extremities of the lower base.

Ex. 41. The lines which join the middle points of the bases of an isosceles trapezoid to the middle points of the non-parallel sides form, with the sides of the trapezoid, two pairs of equal triangles.

Ex. 42. If lines be drawn from the middle point of the base of an isosceles triangle to the middle points of the equal sides, the triangle is divided into two equal triangles and a parallelogram.

POLYGONS

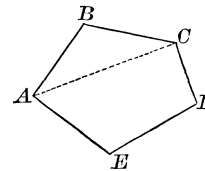
DEFINITIONS

115. We define a *polygon* as a portion of a plane bounded by three or more straight lines; as $ABCDE$.

We call the bounding lines the *sides* of the polygon, and their sum the *perimeter*.

The *angles* of the polygon are the angles EAB , ABC , etc., formed by the adjacent sides; their vertices are called the *vertices* of the polygon.

A *diagonal* of a polygon is a straight line joining any two vertices which are not consecutive; as AC .



116. Polygons are named with reference to the number of their sides, as follows:

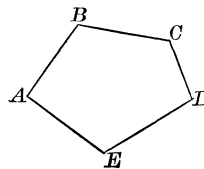
NO. OF SIDES.	DESIGNATION.	NO. OF SIDES.	DESIGNATION.
3	Triangle.	8	Octagon.
4	Quadrilateral.	9	Enneagon.
5	Pentagon.	10	Decagon.
6	Hexagon.	11	Hendecagon.
7	Heptagon.	12	Dodecagon.

117. An *equilateral* polygon is a polygon all of whose sides are equal.

An *equiangular* polygon is a polygon all of whose angles are equal.

118. A polygon is called *convex* when no side, if extended, will enter its surface; as $ABCDE$; in such a case, each angle of the polygon is less than two right angles.

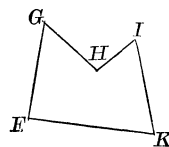
Every polygon considered hereafter will be understood to be convex, unless the contrary is stated.



119. A polygon is called *concave* when at least two of its sides, if extended, will enter its surface; as $FGHIK$.

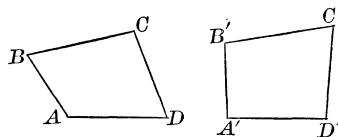
In such a case, at least one angle of the polygon is greater than two right angles.

Thus, in polygon $FGHIK$, the interior angle GHI is greater than two right angles; such an angle is called *re-entrant*.



120. We call two polygons *mutually equilateral* when the sides of one are equal respectively to the sides of the other, when taken in the same order.

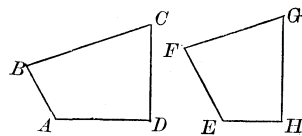
Thus, polygons $ABCD$ and $A'B'C'D'$ are mutually equilateral if $AB = A'B'$, $BC = B'C'$, $CD = C'D'$, and $DA = D'A'$.



121. We call two polygons *mutually equiangular* when the angles of one are equal respectively to the angles of the other, when taken in the same order.

Thus, polygons $EFGH$ and $E'F'G'H'$ are mutually equiangular if

$\angle A = \angle E$, $\angle B = \angle F$, $\angle C = \angle G$, and $\angle D = \angle H$.



122. In polygons which are mutually equilateral or mutually equiangular, sides or angles which are similarly placed are called *homologous*.

123. *If two polygons are both mutually equilateral and mutually equiangular, they are equal.*

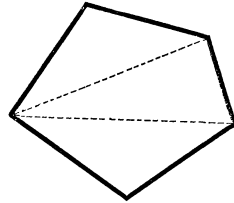
For they can evidently be applied one to the other so as to coincide throughout.

124. *Two polygons are equal when they are composed of the same number of triangles, equal each to each, and similarly placed.*

For they can evidently be applied one to the other so as to coincide throughout.

PROP. XXXIX. THEOREM

125. *The sum of the angles of any polygon is equal to two right angles taken as many times, less two, as the polygon has sides.*



Draw any polygon. We then have :

Given a polygon of n sides.

To Prove the sum of its $\angle$ s equal to $n - 2$ times two rt. $\angle$ s.

Proof. 1. The polygon may be divided into $n - 2$ $\triangle$ s by drawing diagonals from one of its vertices.

2. The sum of the $\angle$ s of the polygon is equal to the sum of the $\angle$ s of the $\triangle$ s.

3. Now the sum of the $\angle$ s of each $\triangle$ is two rt. $\angle$ s.

[The sum of the $\angle$ s of any $\triangle$ is equal to two rt. $\angle$ s.] (§ 84)

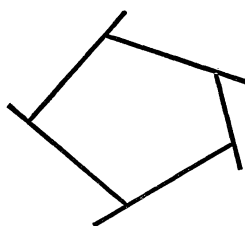
4. Then, the sum of the $\angle$ s of the polygon is $n - 2$ times two rt. $\angle$ s.

126. We have $(n - 2) \times 2$ rt. $\angle$ s equal to $2n$ rt. $\angle$ s $- 4$ rt. $\angle$ s.

Then, *the sum of the angles of any polygon equals twice as many right angles as the polygon has sides, less four right angles.*

PROP. XL. THEOREM

127. *If the sides of any polygon be produced so as to form an exterior angle at each vertex, the sum of these exterior angles is equal to four right angles.*



Draw a polygon in accordance with the statement of the proposition. We then have:

Given a polygon of n sides with its sides extended so as to form an ext. $\angle$ at each vertex.

To Prove the sum of the ext. $\angle$ s equal to 4 rt. $\angle$ s.

Proof. 1. The sum of the ext. and int. $\angle$ s at each vertex is two rt. $\angle$ s.

[If two adj. $\angle$ s have their ext. sides in the same str. line, their sum is equal to two rt. $\angle$ s.] (§ 17)

2. Hence, the sum of *all* the ext. and int. $\angle$ s is $2n$ rt. $\angle$ s.

3. But the sum of the int. $\angle$ s alone is $2n$ rt. $\angle$ s $- 4$ rt. $\angle$ s.

[The sum of the $\angle$ s of any polygon equals twice as many rt. $\angle$ s as the polygon has sides, less 4 rt. $\angle$ s.] (§ 126)

4. Whence, the sum of the ext. $\angle$ s is 4 rt. $\angle$ s.

Ex. 43. Find the sum of the angles of a quadrilateral; of a pentagon; of a hexagon; of an octagon.

Ex. 44. An exterior angle of an equiangular polygon is one-fifth of a right angle. How many sides has the polygon?

Ex. 45. How many sides has a polygon each of whose interior angles is eight-fifths of a right angle?

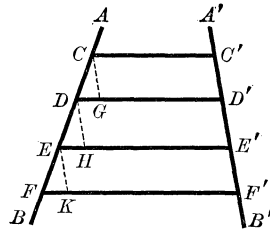
Ex. 46. The difference between two consecutive angles of a parallelogram is 75° ; find the angles of the parallelogram. Find the angles when this difference is M° .

Ex. 47. How many sides has the polygon the sum of whose interior angles is 14 right angles? Find the magnitude of each angle if the polygon is equiangular.

MISCELLANEOUS THEOREMS

PROP. XLI. THEOREM

128. *If a series of parallels, cutting two straight lines, intercept equal distances on one of these lines, they also intercept equal distances on the other.*



Draw a series of parallels (§ 70). Draw lines AB and $A'B'$ cutting these $\parallel$ s at C, D, E, F , and C', D', E', F' , respectively. We then have :

Given lines AB and $A'B'$ cut by $\parallel$ s CC', DD', EE' , and FF' at points C, D, E, F , and C', D', E', F' , respectively, so that

$$CD = DE = EF.$$

To Prove

$$C'D' = D'E' = E'F'.$$

Proof. 1. Draw lines CG, DH , and $EK \parallel A'B'$, meeting DD', EE' , and FF' at points G, H , and K , respectively.

2. In $\triangle CDG, DEH, EFK$, by hyp., $CD = DE = EF$.

3. Again, lines CG, DH , and EK are $\parallel$ to each other.

[Two str. lines $\parallel$ to the same str. line are $\parallel$ to each other.] (§ 71)

4. Then, $\angle DCG = \angle EDH = \angle FEK$.

[If two $\parallel$ lines are cut by a transversal, the corresp. $\angle$ s are equal.]

(§ 76)

5. Also, $\angle CDG = \angle DEH = \angle EFK$.

[If two $\parallel$ lines are cut by a transversal, the corresp. $\angle$ s are equal.]

(§ 76)

6. Then, $\triangle CDG = \triangle DEH = \triangle EFK$.

[Two $\triangle$ are equal when a side and two adj. $\angle$ s of one are equal respectively to a side and two adj. $\angle$ s of the other.] (§ 49)

7. Then, $CG = DH = EK$.

[In equal figures, the homologous parts are equal.] (§ 48)

8. But, $CG = C'D'$, $DH = D'E'$, and $EK = E'F'$.

[In any $\square$, the opposite sides are equal.] (§ 104)

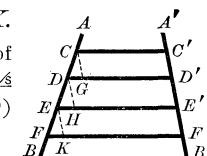
9. Then, $C'D' = D'E' = E'F'$.

[Things which are equal to equal things, are equal to each other.] (Ax. 1)

129. *The line which bisects one side of a triangle, and is parallel to another side, bisects also the third side.*

If line DE is $\parallel$ side BC of $\triangle ABC$, and bisects side AB , it also bisects AC .

This is the particular case in the figure of Prop. XLI where CG coincides with $C'D'$.



130. In the figure of Prop. XLI, $CC'E'E$ is a trapezoid, and DD' bisects its sides CE and $C'E'$.

Hence, *the line which is parallel to the bases of a trapezoid and bisects one of the non-parallel sides, bisects the other also.*

PROP. XLII. THEOREM

131. *The line joining the middle points of two sides of a triangle is parallel to the third side, and equal to one-half of it.*

Draw any $\triangle ABC$. Through D and E , middle points of AB and AC , respectively, draw line DE . We then have :

Given line DE joining middle points of sides AB and AC , respectively, of $\triangle ABC$.

To Prove $DE \parallel BC$, and $DE = \frac{1}{2} BC$.

Proof. 1. A line from $D \parallel BC$ will bisect AC .

[The line which bisects one side of a $\triangle$, and is $\parallel$ to another side, bisects also the third side.] (§ 129)

2. Then it must coincide with DE , and $DE \parallel BC$.

[But one str. line can be drawn between two points.] (Ax. 6)

3. Draw line $EF \parallel AB$, meeting side BC at F .

4. In $\triangle ADE$ and EFC , by hyp., $AE = EC$.

5. Also, $\angle A = \angle CEF$, and $\angle AED = \angle C$.

[If two $\parallel$ s are cut by a transversal, the corresp. $\angle$ s are equal.] (§ 76)

6. Then, $\triangle ADE = \triangle EFC$.

[Two $\triangle$ s are equal when they have a side and two adjacent $\angle$ s of one equal respectively to a side and two adjacent $\angle$ s of the other.] (§ 49)

7. Then, $DE = FC$.

[In equal figures, the homologous parts are equal.] (§ 48)

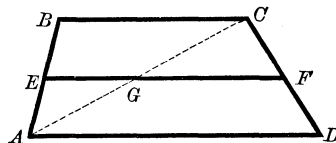
8. But, $DE = BF$.

[In any $\square$, the opposite sides are equal.] (§ 104)

9. Then, $DE = \frac{1}{2} BC$.

PROP. XLIII. THEOREM

132. *The line joining the middle points of the non-parallel sides of a trapezoid is parallel to the bases, and equal to one-half their sum.*



Draw trapezoid $ABCD$, BC and AD being $\parallel$ sides. Let E , F be middle points of AB and CD , respectively. Draw line EF . We then have :

Given line EF joining middle points of non- $\parallel$ sides AB and CD , respectively, of trapezoid $ABCD$.

To Prove $EF \parallel$ to AD and BC , and $EF = \frac{1}{2} (AD + BC)$.

Proof. 1. A line from $E \parallel$ to AD and BC will bisect CD .

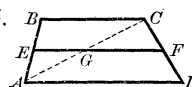
[The line which is $\parallel$ to the bases of a trapezoid, and bisects one of the non- $\parallel$ sides, bisects the other also.] (§ 130)

2. Then it coincides with EF , and EF is $\parallel$ to AD and BC .

[But one str. line can be drawn between two points.] (Ax. 6)

3. Draw diagonal AC , intersecting EF at G .

4. Since EG bisects side AB of $\triangle ABC$, and is $\parallel BC$, it bisects side AC .



[The line which bisects one side of a $\triangle$, and is $\parallel$ to another side, bisects also the third side.] (§ 129)

5. Then, $EG = \frac{1}{2} BC$. (1)

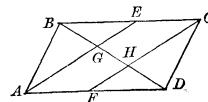
[The line joining the middle points of two sides of a $\triangle$ is $\parallel$ to the third side, and equal to one-half of it.] (§ 131)

6. In like manner, $GF = \frac{1}{2} AD$. (2)

7. Adding (1) and (2), $EF = \frac{1}{2} BC + \frac{1}{2} AD = \frac{1}{2} (BC + AD)$.

Ex. 48. A straight line is drawn from the vertex of a triangle to any point in the base. Prove that this line is bisected by the line joining the middle points of the sides of the triangle.

Ex. 49. E and F are middle points of sides BC and AD , respectively, of the parallelogram $ABCD$. Draw lines AE and CF . Prove that these lines trisect the diagonal joining the other two vertices of the parallelogram.



PROP. XLIV. THEOREM

133. *The bisectors of the angles of a triangle intersect at a common point.*

Draw any $\triangle ABC$. Let a , b , c , be the bisectors of $\angle A$, B , and C , respectively. We then have :

Given a , b , and c , the bisectors of the angles of $\triangle ABC$.

To Prove that a , b , and c intersect at a common point.

Proof. 1. Let a and b intersect at O .

2. Since O is in bisector a , it is equally distant from sides AB and AC .

[Any point in the bisector of an $\angle$ is equally distant from the sides of the $\angle$.] (§ 98)

3. Since O is in bisector b , it is equally distant from sides AB and BC .

4. Then O is equally distant from sides AC and BC , and therefore lies in bisector c .

[Every point which is within an $\angle$, and equally distant from its sides, lies in the bisector of the $\angle$.] (§ 99)

5. Then, a , b , and c intersect in the common point O .

134. It follows from § 133 that

The point of intersection of the bisectors of the angles of a triangle is equally distant from the sides of the triangle.

Ex. 50. The lines joining the middle points of the bases of an isosceles trapezoid to the middle points of the non-parallel sides form a rhombus.

Ex. 51. If through the vertex A , of isosceles triangle ABC , a parallel to BC be drawn, and through the middle points of AB and AC parallels, respectively, to AC and AB , a triangle is formed equal to the given triangle, and its vertex will lie in the base of the given triangle.

Ex. 52. Two sides of a quadrilateral are parallel. The other sides are equal, but not parallel. Prove the opposite angles supplementary.

PROP. XLV. THEOREM

135. *The perpendiculars erected at the middle points of the sides of a triangle intersect at a common point.*

Draw any $\triangle ABC$; at D , E , and F , middle points of BC , CA , and AB , respectively, of $\triangle ABC$, draw lines DG , EH , and $FK \perp$ to BC , CA , and AB , respectively. We now have :

Given DG , EH , and FK , the $\perp$ s erected at the middle points of the sides of $\triangle ABC$.

To Prove that DG , EH , and FK intersect at a common point.

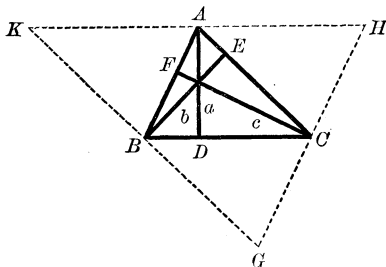
(If DG and EH intersect at O , O is equally distant from B and C , by § 54; and also from A and C .)

136. It follows from § 135 that

The point of intersection of the perpendiculars erected at the middle points of the sides of a triangle, is equally distant from the vertices of the triangle.

PROP. XLVI. THEOREM

137. *The perpendiculars from the vertices of a triangle to the opposite sides intersect at a common point.*



Draw any $\triangle ABC$. From A , B , and C draw lines AD (a), BE (b), and CF (c) $\perp$ to BC , CA , and AB , respectively. We then have :

Given a , b , and c , the three $\perp$ s from the vertices of $\triangle ABC$ to the opposite sides.

To Prove that a , b , and c intersect at a common point.

Proof. 1. Through A , B , and C draw lines HK , KG , and $GH \parallel BC$, CA , and AB , respectively, forming $\triangle GHK$.

2. Since a is $\perp BC$, it is also $\perp HK$.

[A str. line $\perp$ to one of two $\parallel$ s is $\perp$ to the other.] (§ 72)

3. Since, by cons., $ABCH$ and $ACBK$ are $\square$ s,

$$AH = BC \text{ and } AK = BC.$$

[In any $\square$, the opposite sides are equal.] (§ 104)

4. Then, $AH = AK$.

[Things which are equal to the same thing, are equal to each other.] (Ax. 1)

5. Then, a is $\perp HK$ at its middle point.

6. In like manner, b and c are $\perp$ to KG and GH , respectively, at their middle points.

7. Since a , b , and c are $\perp$ to the sides of $\triangle GHK$ at their middle points, they intersect at a common point.

[The $\perp$ s erected at the middle points of the sides of a $\triangle$ intersect at a common point.] (§ 135)

138. Def. A *median* of a triangle is a line drawn from any vertex to the middle point of the opposite side.

PROP. XLVII. THEOREM

139. *The line drawn from any vertex of a triangle through the point of intersection of medians drawn from the other two vertices, is itself a median.*

Draw $\triangle ABC$; draw medians BE and AD intersecting at O ; through O draw line CO meeting AB at F . We then have:

Given AD and BE , two medians of $\triangle ABC$ intersecting at O .

To Prove CF also a median, or $AF = BF$.

Proof. 1. Draw line $AG \parallel BE$, meeting CF prolonged at G ; also, line BG .

2. Since EO is $\parallel$ to side AG of $\triangle ACG$, and bisects side CA , O is the middle point of CG .

[The line which bisects one side of a $\triangle$, and is $\parallel$ to another side, bisects also the third side.] (§ 129)

3. Since O is the middle point of CG , and D of CB , OD is $\parallel$ to side BG of $\triangle BCG$.

[The line joining the middle points of two sides of a $\triangle$ is $\parallel$ to the third side.] (§ 131)

4. Since $AG \parallel BE$, and $AO \parallel BG$, $AOBG$ is a $\square$, and $AF = BF$.

[The diagonals of a $\square$ bisect each other.] (§ 108)

140. In the figure of Prop. XLVII, $OF = FG$. (§ 108)

Since O is the middle point of CG , $OG = OC$.

Then, $OF = \frac{1}{2} OG = \frac{1}{2} OC$; and hence $OC = \frac{2}{3} CF$.

In like manner, $OA = \frac{2}{3} AD$, and $OB = \frac{2}{3} BE$.

That is, *the point of intersection of the medians of a triangle lies two-thirds the way from each vertex to the opposite side.*

141. As an aid in the solution of original exercises, the following list of certain principles proved in Book I will be found useful.

Two triangles are equal :

When two sides and the included angle of one are equal respectively to two sides and the included angle of the other (§ 46).

When a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other (§ 49).

When the three sides of one are equal respectively to the three sides of the other (§ 52).

Two right triangles are equal :

When the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other (§ 60).

When the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other (§ 61).

When a leg and an acute angle of one are equal respectively to a leg and the homologous acute angle of the other (§ 89).

Two straight lines are equal :

When they are homologous sides of equal triangles (§ 48).

When they are opposite equal angles in a triangle (§ 90).

When they are opposite sides of a parallelogram (§ 104).

When they are diagonals of a rectangle (§ 112).

Two angles are equal :

When they are the complements or supplements of equal angles (§ 41).

When they are vertical (§ 44).

When they are homologous angles of equal triangles (§ 48).

When they are opposite the equal sides of an isosceles triangle (§ 50).

When they are alternate-interior angles (§ 75).

When they are corresponding angles (§ 76).

When their sides are parallel each to each (§ 81).

When their sides are perpendicular each to each (§ 83).

When they are opposite angles of a parallelogram (§ 104).

Two lines are parallel :

When they are perpendicular to the same line (§ 70).

When they are parallel to the same line (§ 71).

When the alternate-interior angles are equal (§ 78).

When the corresponding angles are equal (§ 79).

When the sum of the interior angles on the same side of the transversal is equal to two right angles (§ 80).

When they are opposite sides of a parallelogram.

When one line is a side of a triangle, and the other joins the middle points of the other two sides (§ 131).

A quadrilateral is a parallelogram :

When the opposite sides are equal (§ 106).

When two sides are equal and parallel (§ 107).

When the diagonals bisect each other (§ 109).

One line is greater than another :

When it is the hypotenuse of a right triangle of which the other line is a leg (§ 94).

When it is opposite a greater angle in a triangle (§ 93).

One angle is greater than another :

When it is an exterior angle of a triangle in which the other is an opposite interior angle (§ 86).

When it is opposite a greater side in a triangle (§ 92).

Ex. 53. Is it always possible to find a point equidistant from three given straight lines ?

Ex. 54. Is it possible to find a point equidistant from four given straight lines ?

Ex. 55. If two medians of a triangle intersect the sides at right angles, the triangle is equiangular.

Ex. 56. If a line joining two opposite vertices of a parallelogram bisects the angles at these vertices, the parallelogram is equilateral.

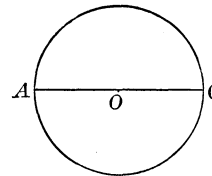
Ex. 57. Prove that lines drawn from two vertices of a triangle and terminating in the opposite sides cannot bisect each other.

Book II

THE CIRCLE

DEFINITIONS

142. A *diameter* of a circle is a straight line drawn through the centre, having its extremities in the circumference.



143. By the definition of § 23

All radii of a circle are equal.

Also, all its diameters are equal, since each is the sum of two radii.

144. *Two circles are equal when their radii are equal.*

For they can evidently be applied one to the other so that their circumferences shall coincide throughout.

145. *The radii of equal circles are equal.* (Converse of § 144.)

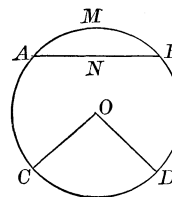
146. *Concentric circles* are circles having the same centre.

147. A *chord* is a straight line joining the extremities of an arc; as *AB*.

The arc is said to be *subtended* by its chord.

Every chord subtends two arcs; thus chord *AB* subtends arcs *AMB* and *ACDB*.

When *the* arc subtended by a chord is spoken of, the one which is less than a semi-circumference is understood, unless the contrary is specified.



A *segment* of a circle is the portion included between an arc and its chord; as *AMBN*.

A *sector* of a circle is the portion included between an arc and the radii drawn to its extremities; as OCD .

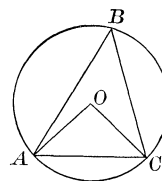
148. A straight line cannot meet a circumference at more than two points.

For by § 64, not more than two equal straight lines can be drawn from a point (in this case, the centre) to a straight line.

149. A *central angle* is an angle whose vertex is at the centre, and whose sides are radii; as AOC .

An *inscribed angle* is an angle whose vertex is on the circumference, and whose sides are chords; as ABC .

An angle is *inscribed in a segment* when its vertex is on the arc of the segment, and its sides pass through the extremities of the subtending chord; thus, angle B is inscribed in segment ABC .

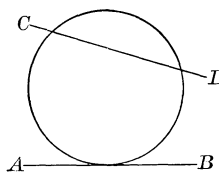


150. A straight line is said to be *tangent to*, or to *touch*, a circle when it has but one point in common with the circumference; as AB .

In such a case, the circle is said to be tangent to the straight line.

The common point is called the *point of contact*.

A *secant* is a straight line which intersects the circumference in two points; as CD .



151. Two circles are said to be *tangent to each other* when they are tangent to the same straight line at the same point.

They are said to be tangent *internally* or *externally* according as one circle lies entirely within or entirely without the other.

A *common tangent* to two circles is a straight line which is tangent to both of them.

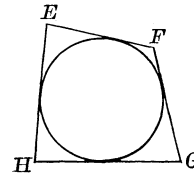
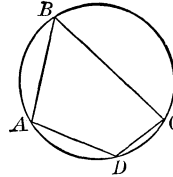
152. A polygon is said to be *inscribed in a circle* when all its vertices lie on the circumference; as $ABCD$.

In this case, the circle is said to be *circumscribed about the polygon*.

A polygon is said to be *inscriptible* when it can be inscribed in a circle.

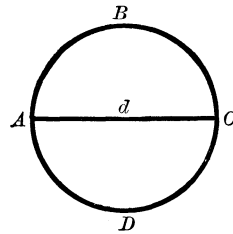
A polygon is said to be *circumscribed about a circle* when all its sides are tangent to the circle; as $EFGH$.

In this case, the circle is said to be *inscribed in the polygon*.



PROP. I. THEOREM

153. Every diameter bisects the circle and its circumference.



Given $AC(d)$ a diameter of $\odot ABCD$.

To Prove that d bisects the $\odot$, and its circumference.

Proof. 1. Superpose segment ABC upon segment ADC , by folding it over about d as an axis.

2. Then, arc ABC will coincide with arc ADC ; for otherwise there would be points of the circumference unequally distant from the centre.

3. Hence, segments ABC and ADC coincide throughout, and are equal.

4. Therefore, d bisects the $\odot$, and its circumference.

154. Defs. A *semi-circumference* is an arc equal to one-half the circumference.

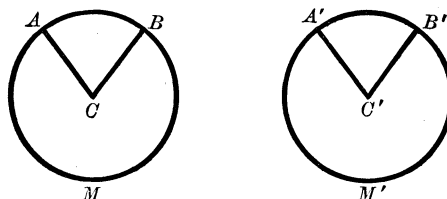
A *quadrant* is an arc equal to one-fourth the circumference.

A *semicircle* is a segment equal to one-half the circle.

Ex. 1. If two non-perpendicular diameters be drawn in the same circle, the perpendiculars from the extremities of the one upon the other are equal.

PROP. II. THEOREM

155. In equal circles, or in the same circle, equal central angles intercept equal arcs on the circumference.



Draw $\odot AMB$ and $\odot A'M'B'$, having centres C and C' , respectively, and same radius. In $\odot AMB$, draw $\angle ACB$; in $\odot A'M'B'$, construct $\angle A'C'B' = \angle ACB$. We then have:

Given ACB and $A'C'B'$ equal central $\angle$ s of equal $\odot AMB$ and $\odot A'M'B'$, respectively.

To Prove arc $AB = \text{arc } A'B'$.

Proof. 1. Superpose sector ABC upon sector $A'B'C'$ in such a way that $\angle C$ shall coincide with its equal $\angle C'$.

2. We have $AC = A'C'$ and $BC = B'C'$. (§ 145)

3. Then, point A will fall at A' , and point B at B' .

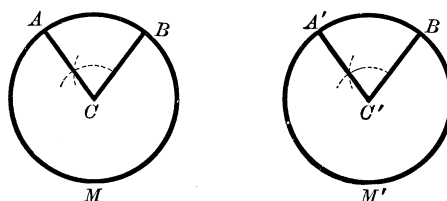
4. Then, arc AB will coincide with arc $A'B'$; for all points of either are equally distant from the centre.

5. Whence, arc $AB = \text{arc } A'B'$.

Ex. 2. Draw lines dividing a circumference into eight equal parts.

PROP. III. THEOREM

156. (Converse of Prop. II.) *In equal circles, or in the same circle, equal arcs are intercepted by equal central angles.*



Draw $\odot AMB$ and $A'M'B'$, having centres C and C' , respectively, and same radius. Take arc $AB = \text{arc } A'B'$, and draw lines AC , BC , $A'C'$, and $B'C'$. We then have:

Given ACB and $A'C'B'$ central $\angle$ s of equal $\odot AMB$ and $A'M'B'$, respectively, and arc $AB = \text{arc } A'B'$.

To Prove $\angle C = \angle C'$.

Proof. 1. Since the $\odot$ are equal, we may superpose $\odot AMB$ upon $\odot A'M'B'$ in such a way that point A shall fall at A' , and centre C at C' .

2. Since arc $AB = \text{arc } A'B'$, point B will fall at B' .

3. Then, radii AC and BC will coincide with radii $A'C'$ and $B'C'$, respectively. (Ax. 6)

4. Then, $\angle C$ will coincide with $\angle C'$; that is, $\angle C = \angle C'$.

157. Note. *In equal circles, or in the same circle,*

1. *The greater of two central angles intercepts the greater arc on the circumference.*

2. *The greater of two arcs is intercepted by the greater central angle.*

PROP. IV. THEOREM

158. *In equal circles, or in the same circle, equal chords subtend equal arcs.*

Draw $\odot AMB$ and $A'M'B'$, having same radius. In $\odot AMB$, draw chord AB ; in $\odot A'M'B'$, draw chord $A'B' = \text{chord } AB$. We then have:

Given, in equal $\odot AMB$, $A'M'B'$, chord $AB = \text{chord } A'B'$.

To Prove arc $AB = \text{arc } A'B'$.

Proof. 1. Draw radii $AC, BC, A'C',$ and $B'C'$.

2. In $\triangle ABC$ and $A'B'C'$, by hyp., $AB = A'B'$.

3. Also, $AC = A'C'$, and $BC = B'C'$. (§ 145)

4. Then, $\triangle ABC = \triangle A'B'C'$. (§ 52)

5. Then, $\angle C = \angle C'$. (?)

6. Then, arc $AB = \text{arc } A'B'$. (§ 155)

Ex. 3. If A, B, C, D are points in succession on a circumference, and arcs $AB, BC,$ and CD are equal, prove chord $AC = \text{chord } BD$.

PROP. V. THEOREM

159. (Converse of Prop. IV.) *In equal circles, or in the same circle, equal arcs are subtended by equal chords.*

Given, in equal $\odot AMB$ and $A'M'B'$, arc $AB = \text{arc } A'B'$; and chords AB and $A'B'$. (Fig. of Prop. IV.)

To Prove chord $AB = \text{chord } A'B'$.

(The $\triangle ABC$ and $A'B'C'$ are equal by § 46.)

PROP. VI. THEOREM

160. *In equal circles, or in the same circle, the greater of two arcs is subtended by the greater chord; each arc being less than a semi-circumference.*

Draw $\odot AMB$ and $A'M'B'$, having same radius, and centres at C and C' , respectively. Take arc $AB > \text{arc } A'B'$, each arc $<$ a semi-circumference. Draw lines AB and $A'B'$. We then have :

Given, in equal $\odot AMB$ and $A'M'B'$, arc $AB > \text{arc } A'B'$, each arc $<$ a semi-circumference, and chords AB and $A'B'$.

To Prove chord $AB > \text{chord } A'B'$.

Proof. 1. Draw radii $AC, BC, A'C',$ and $B'C'$.

2. In $\triangle ABC$ and $A'B'C'$, $AC = A'C'$, and $BC = B'C'$. (?)

3. Since, by hyp., arc $AB > \text{arc } A'B'$,

$$\angle C > \angle C'. \quad (\S 157, 2)$$

4. Then, chord $AB > \text{chord } A'B'$. (§ 100)

PROP. VII. THEOREM

161. (Converse of Prop. VI.) *In equal circles, or in the same circle, the greater of two chords subtends the greater arc; each arc being less than a semi-circumference.*

Given, in equal $\odot AMB$ and $A'M'B'$, chord $AB >$ chord $A'B'$, arcs AB and $A'B'$ being $<$ a semi-circumference. (Fig. of Prop. VI.)

To Prove arc $AB >$ arc $A'B'$.

(We have $\angle C > \angle C'$, by § 101; then use § 157, 1.)

Ex. 4. A central angle AOB is bisected by a radius OC . Let A' , C' , B' be the middle points of OA , OC , and OB , respectively. Prove the length of the broken line $A'C'B'$ equal to the length of the chord AC .

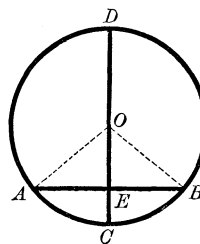
PROP. VIII. THEOREM

162. *The diameter perpendicular to a chord bisects the chord and its subtended arcs.*

Draw $\odot ABD$ with centre at O ; draw diameter DC ; through point E on OC draw chord $AB \perp DC$. We then have:

Given, in $\odot ABD$, diameter $CD \perp$ chord AB .

To Prove that CD bisects chord AB , and arcs ACB and ADB .



Proof. 1. Let O be centre of $\odot$; draw radii OA and OB .

2. Since $OA = OB$, $\triangle OAB$ is isosceles.

3. Then, CD bisects AB , and $\angle AOB$. (§ 91)

4. Since $\angle AOC = \angle BOC$, arc $AC =$ arc BC . (§ 155)

5. Again, $\angle AOD = \angle BOD$. (§ 41, 2)

6. Then, arc $AD =$ arc BD . (?)

163. It follows from the above that *the perpendicular erected at the middle point of a chord passes through the centre of the circle, and bisects the arcs subtended by the chord.*

PROP. IX. THEOREM

164. *In the same circle, or in equal circles, equal chords are equally distant from the centre.*

Draw a $\odot$ with centre at O ; draw equal chords AB and CD . Draw lines OE and $OF \perp AB$ and CD , respectively, and meeting these lines at E and F , respectively. We then have:

Given AB and CD equal chords of $\odot ABC$, whose centre is O , and lines OE and $OF \perp AB$ and CD , respectively.

To Prove $OE = OF$.

Proof. 1. Draw radii OA and OC .

2. In rt. $\triangle OAE$ and OCF , $OA = OC$. (?)

3. E is the middle point of AB , and F of CD . (§ 162)

4. Then, $AE = CF$;

for they are halves of equal chords AB and CD , respectively.

5. Then, $\triangle OAE = \triangle OCF$. (?)

6. Then, $OE = OF$. (?)

Ex. 5. If two equal chords, not diameters, intersect within a circle, and perpendiculars be drawn from the centre to these chords, the line joining the centre of the circle with the intersection of these chords is a bisector of the angle formed by the perpendiculars.

Ex. 6. If two circles are concentric, and a chord of the greater is a secant of the smaller, then the portions of the chord between the two circumferences are equal.

Ex. 7. The line which bisects a chord of a circle and its subtended arc will, if extended, pass through the centre of the circle.

PROP. X. THEOREM

165. (Converse of Prop. IX.) *In the same circle, or in equal circles, chords equally distant from the centre are equal.*

Given O the centre of $\odot ABC$, and AB and CD chords equally distant from O . (Fig. of Prop. IX.)

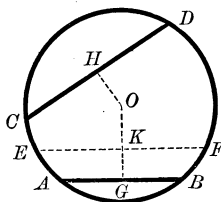
To Prove chord $AB =$ chord CD .

(The rt. $\triangle OAE$ and OCF are equal (?), and $AE = CF$; E is the middle point of AB , and F of CD .)

PROP. XI. THEOREM

166. *In the same circle, or in equal circles, if two chords are unequally distant from the centre, the more remote is the less.*

Draw figure in accordance with the statement. We then have:



Given O the centre of $\odot ABC$, and chord AB more remote from O than chord CD .

To Prove chord $AB < \text{chord } CD$.

Proof. 1. Draw lines $OG \perp AB$ and $OH \perp CD$; on OG take $OK = OH$, and through K draw chord $EF \perp OK$.

2. Then, chord $EF = \text{chord } CD$. (§ 165)

3. Now, chord $AB \parallel \text{chord } EF$. (§ 70)

4. Then, arc AB must be $<$ arc EF .

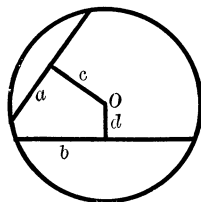
5. Then, chord $AB < \text{chord } EF$. (§ 160)

6. That is, chord $AB < \text{chord } CD$.

167. By § 166, a diameter is greater than any other chord.

PROP. XII. THEOREM

168. (Converse of Prop. XI.) *In the same circle, or in equal circles, the less of two chords is at the greater distance from the centre.*



Draw $\odot$ with centre at O ; draw chords a and b , a being $<$ b , their $\perp$ distances from O being c and d , respectively. We then have:

Given, in $\odot$ whose centre is O , chords a and b , respectively, at distances c and d from O , and chord $a < \text{chord } b$.

To Prove $c > d$.

- Proof.** 1. If c is $< d$, chord a will be $>$ chord b . (§ 166)
 2. If $c = d$, chord a will equal chord b . (§ 165)
 3. Both of these results are contrary to the hyp. that chord a is $<$ chord b .
 4. Then, $c > d$.

PROP. XIII. THEOREM

169. *A straight line perpendicular to a radius of a circle at its extremity is tangent to the circle.*

Draw a $\odot$ with centre at O , and radius OC . Through point C draw line $AB \perp OC$. We then have:

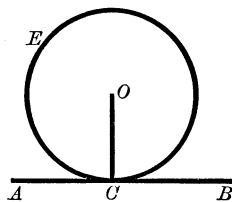
Given line $AB \perp$ radius OC of $\odot O$ at C .

To Prove AB tangent to the $\odot$.

- Proof.** 1. Let D be any point of AB except C .
 2. Draw line OD ; then, $OD > OC$. (?)
 3. Therefore, point D lies without the $\odot$.
 4. Then, every point of AB except C lies without the $\odot$, and AB is tangent to the $\odot$. (§ 150)

PROP. XIV. THEOREM

170. (Converse of Prop. XIII.) *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*



Draw $\odot$ with centre at O . Draw line AB tangent to the $\odot$ at C . Draw line OC . We then have:

Given line AB tangent to $\odot EC$ at C , and radius OC .

To Prove $OC \perp AB$.

(OC is the shortest line from O to AB .)

171. It follows from § 170 that a line perpendicular to a tangent at its point of contact passes through the centre of the circle.

PROP. XV. THEOREM

172. Two parallels intercept equal arcs on a circumference.

Case I. When one line is a tangent and the other a secant.

Draw a $\odot$. Through E , any point on the circumference, draw diameter EF , and tangent $AB \perp EF$ (§ 169). Draw secant $CD \parallel AB$, intersecting the circumference at C and D . We then have:

Given AB a tangent to $\odot CED$ at E , and CD a secant $\parallel AB$, intersecting the circumference at C and D .

To Prove $\text{arc } CE = \text{arc } DE$.

Proof. 1. Draw diameter EF ; then $EF \perp AB$. (§ 170)

2. Then, $EF \perp CD$. (?)

3. Then, $\text{arc } CE = \text{arc } DE$. (§ 162)

Case II. When both lines are secants.

Draw a $\odot$. Draw $\parallel$ secants, AB and CD , intersecting the circumference at A and B , and C and D , respectively. We then have:

Given, in $\odot ABC$, AB and $CD \parallel$ secants, intersecting the circumference at A and B , and C and D , respectively.

To Prove $\text{arc } AC = \text{arc } BD$.

Proof. 1. Draw tangent $EF \parallel AB$, touching the $\odot$ at G .

2. Then, $EF \parallel CD$. (?)

3. By Case I, $\text{arc } AG = \text{arc } BG$, (1)

and $\text{arc } CG = \text{arc } DG$. (2)

4. Subtracting (2) from (1),

$$\text{arc } AG - \text{arc } CG = \text{arc } BG - \text{arc } DG,$$

or $\text{arc } AC = \text{arc } BD$.

Case III. When both lines are tangents.

In $\odot EGF$, draw $\parallel$ tangents, AB and CD , touching $\odot$ at E and F , respectively. We then have :

Given, in $\odot EGF$, AB and $CD \parallel$ tangents, touching the $\odot$ at E and F , respectively.

To Prove $\text{arc } EGF = \text{arc } EHF$.

(Draw secant $GH \parallel AB$, and apply Case I.)

173. It follows from § 172 that *the straight line joining the points of contact of two parallel tangents is a diameter*.

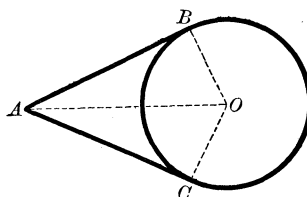
Ex. 8. The diameters of two concentric circles are 11 and 15 inches, respectively. Can a tangent to either circle be drawn through a point 5 inches from the common centre ? Why ?

Ex. 9. The lines joining the extremities of two unequal parallel chords form, with the chords, an isosceles trapezoid. Is the centre within or without the trapezoid ?

Ex. 10. If two equal chords, AB and CD , of a circle are parallel, the points A , B , D , and C being in succession on the circumference, the chords AC and BD are equal and the figure formed is a rectangle. Can the centre be without the rectangle ?

PROP. XVI. THEOREM

174. *The tangents to a circle from an outside point are equal.*



Draw $\odot$ with centre at O . Draw radii OB and OC , and tangents $\perp OB$ and OC , respectively, intersecting at A . We then have :

Given AB and AC tangent at points B and C , respectively, to $\odot BC$ whose centre is O .

To Prove $AB = AC$.

(Rt. $\triangle OAB$ and OAC are equal by § 61.)

175. From equal $\triangle OAB$ and OAC (Fig. of Prop. XVI),
 $\angle OAB = \angle OAC$ and $\angle AOB = \angle AOC$.

Then, *the line joining the centre of a circle to the point of intersection of two tangents makes equal angles with the tangents, and also with the radii drawn to the points of contact.*

Ex. 11. The line through the middle points of two parallel chords passes through the centre of the circle.

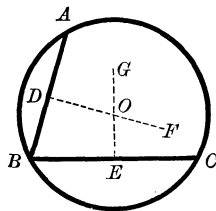
Ex. 12. If two equal circles intersect and parallel lines be drawn through the points of intersection, the portion cut from the first line by one circumference is equal to the portion cut from the second by the other circumference.

Ex. 13. The angle formed by two tangents to a circle is the supplement of the angle formed by joining the centre to the points of tangency.

Ex. 14. Can a trapezoid be constructed whose sides are 8, 10, 11, 12, respectively? Give method showing the result. Can such a trapezoid be inscribed in a circle?

PROP. XVII. THEOREM

176. *Through three points, not in the same straight line, a circumference can be drawn, and but one.*



Given points A , B , and C , not in the same straight line.

To Prove that a circumference can be drawn through A , B , and C , and but one.

Proof. 1. Draw lines AB and BC , and lines DF and $EG \perp AB$ and BC , respectively, at their middle points, meeting at O .

2. Then O is equally distant from A , B , and C . (§ 136)

3. Hence, a circumference described with O as a centre and OA as a radius will pass through A , B , and C .

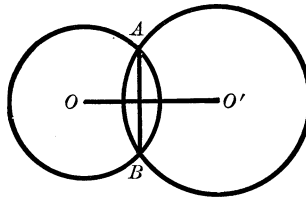
4. Again, the centre of any circumference drawn through A , B , and C must be in each of the $\perp$ DF and EG . (§ 56)

5. Then as DF and EG intersect in but one point, only one circumference can be drawn through A , B , and C .

177. *Two circumferences can intersect in but two points; for if they had three common points, they would have the same centre, and coincide throughout.*

PROP. XVIII. THEOREM

178. *If two circumferences intersect, the straight line joining their centres bisects their common chord at right angles.*



Draw $\odot$, with centres at O and O' , whose circumferences intersect at A and B . Draw lines AB and OO' . We then have:

Given O and O' the centres of two $\odot$, whose circumferences intersect at A and B , and lines OO' and AB .

To Prove that OO' bisects AB at rt. $\angle$.

(Use § 56.)

PROP. XIX. THEOREM

179. *If two circles are tangent to each other, the straight line joining their centres passes through their point of contact.*

Draw two $\odot$, whose centres are at O and O' , tangent to line AB at A . We then have:

Given O and O' the centres of two $\odot$, which are tangent to line AB at A .

To Prove that str. line joining O and O' passes through A .

(Draw radii OA and $O'A$; these lines are $\perp$ AB , and $OA O'$ is a str. line by § 20. Then use Ax. 6.)

Ex. 15. If two circumferences intersect, the distance between their centres is greater than the difference of their radii.

Ex. 16. If a tangent has its extremities in two parallel tangents to the same circle, the angle which is formed by the lines joining the centre of the circle to the extremities of the first tangent is a right angle.

[Draw radii to the points of tangency.]

MEASUREMENT

180. The *ratio* of one magnitude to another of the same kind is the quotient of the first divided by the second.

Thus, if a and b are quantities of the same kind, the ratio of a to b is $\frac{a}{b}$.

We *measure* a magnitude by finding its ratio to another magnitude of the same kind, called the *unit of measure*.

If the quotient can be obtained exactly as an integer or fraction, we call it the *numerical measure* of the magnitude.

181. Two magnitudes of the same kind are said to be *commensurable* when a unit of measure (called a *common measure*) is contained an *integral* number of times in each.

Thus, two lines whose lengths are $2\frac{3}{4}$ and $3\frac{1}{5}$ inches are commensurable; for the common measure $\frac{1}{20}$ inch is contained an integral number of times in each; *i.e.* 55 times in the first line, and 76 times in the second.

Two magnitudes of the same kind are said to be *incommensurable* when no magnitude of the same kind can be found which is contained an integral number of times in each.

For example, if AB and CD are two lines such that

$$\frac{AB}{CD} = \sqrt{2};$$

since $\sqrt{2}$ can only be obtained *approximately* as a decimal, no line, *however small*, can be found which is contained an integral number of times in each line, and AB and CD are incommensurable.

182. A magnitude which is incommensurable with respect to the unit has no exact numerical measure (§ 180).

Still if CD is the unit of measure, and $\frac{AB}{CD} = \sqrt{2}$, we shall call $\sqrt{2}$ the numerical measure of AB .

183. It is evident from the above that the ratio of two magnitudes of the same kind, whether commensurable or incommensurable, is equal to the ratio of their numerical measures when referred to a common unit.

THE METHOD OF LIMITS

184. We define a *variable* as a quantity which, under the conditions imposed upon it, may have an indefinitely great number of *different values*.

We define a *constant* as a quantity which remains unchanged throughout the same discussion.

185. A *limit* of a variable is a constant quantity, the difference between which and the variable may be made less than any assigned quantity, however small.

In other words, a limit of a variable is a fixed quantity to which the variable approaches indefinitely near.

186. Suppose, for example, that a point moves from A towards B under the condition that it shall move, during successive equal intervals of time, first from A to C , halfway between A and B ; then to D , halfway between C and B ; then to E , halfway between D and B ; and so on indefinitely.

In this case, the distance between the moving point and B can be made less than any assigned distance, however small.

Then, the distance from A to the moving point is a variable which approaches the constant distance AB as a limit.

Again, the distance from the moving point to B is a variable which approaches the limit 0.

As another illustration, consider the series

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots,$$

where each term after the first is one-half the preceding.

In this case, by taking terms enough, the last term may be made less than any assigned number, however small.

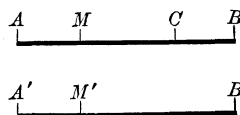
Then, the last term of the series is a variable which approaches the limit 0 when the number of terms is indefinitely increased.

Again, the sum of the first two terms is $1\frac{1}{2}$;
 the sum of the first three terms is $1\frac{3}{4}$;
 the sum of the first four terms is $1\frac{7}{8}$; etc.

In this case, by taking terms enough, the sum of the terms may be made to differ from 2 by less than any assigned number, however small.

Then, the sum of the terms of the series is a variable which approaches the limit 2 when the number of terms is indefinitely increased.

187. The Theorem of Limits. *If two variables are always equal, and each approaches a limit, the limits are equal.*



Given AM and $A'M'$ two variables, which are always equal, and approach the limits AB and $A'B'$, respectively.

To Prove $AB = A'B'$.

- Proof.** 1. If possible, suppose $AB > A'B'$.
 2. Take, on AB , AC equal to $A'B'$.
 3. Then, variable AM may assume values $> AC$, while variable $A'M'$ is restricted to values $< AC$.
 4. This is contrary to the hypothesis that the variables are always equal; then, AB cannot be $> A'B'$.
 5. In like manner, we may prove that AB cannot be $< A'B'$.
 6. Since AB cannot be $>$, nor $< A'B'$, we have $AB = A'B'$.

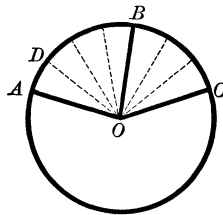
Note. In the above demonstration, we have supposed AM and $A'M'$ to be *increasing* variables; the theorem may be proved in a similar manner if AM and $A'M'$ are *decreasing* variables.

MEASUREMENT OF ANGLES

PROP. XX. THEOREM

188. *In the same circle, or in equal circles, two central angles are in the same ratio as their intercepted arcs.*

Case I. *When the arcs are commensurable (§ 181).*



Draw figure in accordance with the statement. We then have :

Given, in $\odot ABC$, $\angle AOB$ and $\angle BOC$ central $\angle$ s intercepting commensurable arcs AB and BC , respectively.

To Prove
$$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$

Proof. 1. By hyp., arcs AB and BC are commensurable; let arc AD be a common measure of arcs AB and BC , and suppose it to be contained 4 times in arc AB , and 3 times in arc BC .

2. Then,
$$\frac{\text{arc } AB}{\text{arc } BC} = \frac{4}{3}. \quad (1)$$

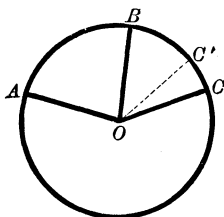
3. Drawing radii to the several points of division of arc AC , $\angle AOB$ will be divided into 4 $\angle$ s, and $\angle BOC$ into 3 $\angle$ s, all of which $\angle$ s are equal. (§ 155)

4. Then,
$$\frac{\angle AOB}{\angle BOC} = \frac{4}{3}. \quad (2)$$

5. From (1) and (2),
$$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}. \quad (?)$$

Note. The theorem may be proved in a similar manner, whatever the number of subdivisions of arcs AB and BC .

Case II. When the arcs are incommensurable (§ 181).



Draw figure in accordance with the statement. We then have :

Given, in $\odot ABC$, AOB and BOC central $\angle$ s intercepting incommensurable arcs AB and BC , respectively.

To Prove

$$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$

Proof. 1. Let arc AB be divided into any number of equal arcs, and let one of these arcs be applied to arc BC as a unit of measure.

2. Since arcs AB and BC are incommensurable, a certain number of equal arcs will extend from B to C' , leaving a remainder $C'C$ less than one of the equal arcs.

3. Draw radius OC' ; since arcs AB and BC' are commensurable,

$$\frac{\angle AOB}{\angle BOC'} = \frac{\text{arc } AB}{\text{arc } BC'}. \quad (\text{Case I})$$

4. Now let the number of subdivisions of arc AB be indefinitely increased.

5. Then the unit of measure will be indefinitely diminished; and the remainder $C'C$, being always less than the unit, will approach the limit 0.

6. Then $\angle BOC'$ will approach the limit $\angle BOC$,
and arc BC' will approach the limit arc BC .

7. Hence, $\frac{\angle AOB}{\angle BOC'}$ will approach the limit $\frac{\angle AOB}{\angle BOC}$,

and $\frac{\text{arc } AB}{\text{arc } BC'}$ will approach the limit $\frac{\text{arc } AB}{\text{arc } BC}$.

8. Now, $\frac{\angle AOB}{\angle BOC'}$ and $\frac{\text{arc } AB}{\text{arc } BC'}$ are two variables which are always equal; and they approach the limits $\frac{\angle AOB}{\angle BOC}$ and $\frac{\text{arc } AB}{\text{arc } BC}$, respectively.

9. By the Theorem of Limits, these limits are equal.

(§ 187)

10. Then, $\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}$.

189. The usual unit of measure for arcs is the *degree*, which is the ninetieth part of a quadrant (§ 154).

The degree of arc is divided into sixty equal parts, called *minutes*, and the minute into sixty equal parts, called *seconds*.

If the sum of two arcs is a quadrant, or 90° , one is called the *complement* of the other; if their sum is a semi-circumference, or 180° , one is called the *supplement* of the other.

190. By § 155, equal central $\angle$ s, in the same $\odot$, intercept equal arcs on the circumference.

Hence, if the angular magnitude about the centre of a $\odot$ be divided into four equal $\angle$ s, each $\angle$ will intercept an arc equal to one-fourth of the circumference.

That is, *a right central angle intercepts a quadrant on the circumference.*

(§ 19)

191. By § 188, a central $\angle$ of n degrees bears the same ratio to a rt. central $\angle$ that its intercepted arc bears to a quadrant.

But a central $\angle$ of n degrees is $\frac{n}{90}$ of a rt. central $\angle$.

Hence, its intercepted arc is $\frac{n}{90}$ of a quadrant, or an arc of n degrees.

The above principle is usually expressed as follows:

A central angle is measured by its intercepted arc.

The above statement signifies simply that the number of angular degrees in a central angle is equal to the number of degrees of arc in its intercepted arc.

PROP. XXI. THEOREM

192. *An inscribed angle is measured by one-half its intercepted arc.*

Case I. *When one side of the angle is a diameter.*

Draw $\odot ABC$ with centre at O . Draw diameter AC and chord AB . We then have :

Given AC a diameter, and AB a chord, of $\odot ABC$.

To Prove that $\angle A$ is measured by $\frac{1}{2}$ arc BC .

Proof. 1. Draw radius OB ; denote $\angle BOC$ by a .

2. We have, $OA = OB$. (?)

3. Then, $\angle B = \angle A$. (§ 50)

4. Since a is an ext. $\angle$ of $\triangle OAB$, $\angle a = \angle A + \angle B$. (§ 85)

5. Then, $\angle a = 2 \angle A$, or $\angle A = \frac{1}{2} \angle a$.

6. But, $\angle a$ is measured by arc BC . (§ 191)

7. Whence, $\angle A$ is measured by $\frac{1}{2}$ arc BC .

Case II. *When the centre is within the angle.*

Draw $\odot ABC$; draw an inscribed $\angle BAC$ with the centre within the $\angle$. We then have :

Given AB and AC chords of $\odot ABC$, and the centre of the $\odot$ within $\angle BAC$.

To Prove that $\angle BAC$ is measured by $\frac{1}{2}$ arc BC .

Proof. 1. Draw diameter AD ; denote $\angle BAD$ by a and $\angle BAC$ by b .

2. By Case I, $\angle a$ is measured by $\frac{1}{2}$ arc BD ,
and $\angle b$ is measured by $\frac{1}{2}$ arc CD .

3. Then, $\angle a + \angle b$ is measured by $\frac{1}{2}$ arc $BD + \frac{1}{2}$ arc CD .

4. Then, $\angle BAC$ is measured by $\frac{1}{2}$ arc BC .

Case III. *When the centre is without the angle.*

Draw $\odot ABC$. Inscribe $\angle BAC$, the centre lying without the $\angle$. Draw diameter AD .

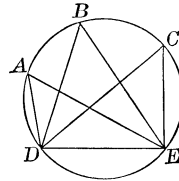
(The proof is left to the pupil.)

193. *Angles inscribed in the same segment are equal.*

For, if A , B , and C are $\angle$ s inscribed in segment AED of $\odot ABC$,

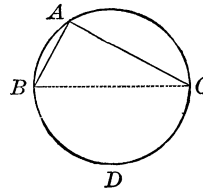
$$\angle A = \angle B = \angle C;$$

for each $\angle$ is measured by $\frac{1}{2}$ arc DE (§ 192).



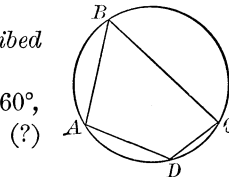
194. *An angle inscribed in a semicircle is a right angle.*

For if BC be a diameter, and AB and AC chords, of $\odot ABD$, $\angle A$ is measured by $\frac{1}{2}$ an arc of 180° , or 90° (§ 192).



195. *The opposite angles of an inscribed quadrilateral are supplementary.*

For their sum is measured by $\frac{1}{2}$ of 360° , or 180° .



Ex. 17. A quadrilateral $ABCD$ is inscribed in a circle. Arc AB is 84° , arc CD is 96° , and $\angle C$ is 80° . Find remaining arcs and angles.

Ex. 18. A quadrilateral $ABCD$ is inscribed in a circle. $\angle B$ is 104° , $\angle C$ is 92° , and arc AB 70° . Find remaining angles of the quadrilateral and the arcs subtended by its sides.

Ex. 19. A regular polygon has seven sides. Find the number of degrees in each angle.

Ex. 20. The median drawn to the hypotenuse of a right triangle divides it into two isosceles triangles.

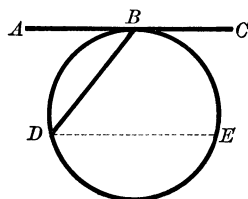
Ex. 21. If angles be inscribed in two segments whose sum is equal to the whole circle, they are supplements and each angle is a supplement of one-half the central angle which its arc subtends.

Ex. 22. A triangle is inscribed in a circle; one of its angles is 74° , and one side of this angle subtends an arc of 42° ; find the remaining angles of the triangle and the subtended arcs.

Ex. 23. One of the equal sides of an isosceles triangle ABC is 30 inches, the vertical angle A is 120° . A circle described on one of the equal sides as a diameter intersects BC at D . Find BD in terms of BC .

PROP. XXII. THEOREM

196. *The angle formed by a tangent and a chord is measured by one-half its intercepted arc.*



Draw a $\odot$. Draw tangent AC , touching the $\odot$ at B , and chord BD making an acute $\angle$ with AB . We then have :

Given AC tangent to $\odot BDE$ at B , and BD a chord.

To Prove that $\angle ABD$ is measured by $\frac{1}{2}$ arc BD .

Proof. 1. Draw chord $DE \parallel AC$; then, $\angle ABD = \angle D$. (§ 75)

2. Now $\angle D$ is measured by $\frac{1}{2}$ arc BE . (§ 192)

3. Also, arc $BE =$ arc BD . (§ 172)

4. Then, $\angle ABD$ is measured by $\frac{1}{2}$ arc BD .

Since $\angle CBD$ is sup. to $\angle ABD$, it is measured by $\frac{1}{2}$ (a circumference — arc BD), or $\frac{1}{2}$ arc BED . (§ 192)

PROP. XXIII. THEOREM

197. *The angle formed by two chords, intersecting within the circumference, is measured by one-half the sum of its intercepted arc, and the arc intercepted by its vertical angle.*

Draw $\odot ABC$. Draw chords AB and CD intersecting within the circumference at E ; denote $\angle AEC$ by a . We then have :

Given, in $\odot ABC$, chords AB and CD intersecting within the circumference at E .

To Prove that $\angle a$ is measured by $\frac{1}{2}$ (arc AC + arc BD).

Proof. 1. Draw chord BC ; since a is an ext. $\angle$ of $\triangle BCE$,
 $\angle a = \angle B + \angle C$. (?)

2. But, $\angle B$ is measured by $\frac{1}{2}$ arc AC ,
 and $\angle C$ is measured by $\frac{1}{2}$ arc BD . (??)

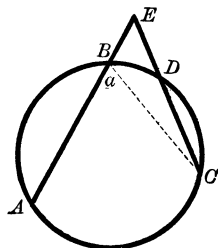
3. Substituting in result of (1),

$$\angle a \text{ is measured by } \frac{1}{2} (\text{arc } AC + \text{arc } BD).$$

Ex. 24. In an inscribed quadrilateral $ABCD$, $\angle A = 75^\circ$, $\angle B = 40^\circ$, arc $BC = 95^\circ$; find the angle formed by the diagonals of the quadrilateral.

PROP. XXIV. THEOREM

198. *The angle formed by two secants, intersecting without the circumference, is measured by one-half the difference of the intercepted arcs.*



Draw $\odot ABC$. Draw secants AE and CE intersecting without the circumference at E , and intersecting the circumference at A and B , and C and D , respectively. We then have :

Given, in $\odot ABC$, secants AE and CE intersecting without the circumference at E , and intersecting the circumference at A and B , and C and D , respectively.

To Prove that $\angle E$ is measured by $\frac{1}{2}$ (arc $AC - \text{arc } BD$).

Proof. 1. Draw chord BC ; denote $\angle ABC$ by a .

2. Since a is an ext. $\angle$ of $\triangle BCE$, $\angle a = \angle E + \angle C$. (??)

3. Then, $\angle E = \angle a - \angle C$.

4. Now $\angle a$ is measured by $\frac{1}{2}$ arc AC ,

and $\angle C$ is measured by $\frac{1}{2}$ arc BD . (??)

5. Then, $\angle E$ is measured by $\frac{1}{2}$ (arc $AC - \text{arc } BD$).

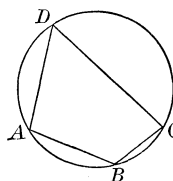
199. *If the opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.*

(Converse of § 195.)

Suppose, in quadrilateral $ABCD$, $\angle A$ sup. to $\angle C$, and $\angle B$ to $\angle D$; and a circumference drawn through A , B , and C . (§ 176)

If $\angle D$ is sup. to $\angle B$, it is measured by $\frac{1}{2}$ arc ABC . (§ 192)

Then, D must lie on the circumference; for if it were within the $\odot$, $\angle D$ would be measured by $\frac{1}{2}$ an arc $> ABC$; and if it were without the $\odot$, $\angle D$ would be measured by $\frac{1}{2}$ an arc $< ABC$. (§§ 197, 198)



Ex. 25. Two equal chords bisect in a circle. What are the angles formed?

Ex. 26. Two chords intersect within a circle, and form an angle of 75° ; one of the intercepted arcs is 60° : find the other.

Ex. 27. Two chords intersect within a circle; one of the angles formed is A° , and one of the intercepted arcs is B° : find the other.

PROP. XXV. THEOREM

200. *The angle formed by a secant and a tangent, or two tangents; is measured by one-half the difference of the intercepted arcs.*

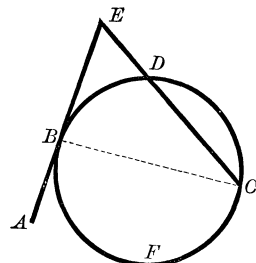


FIG. 1.

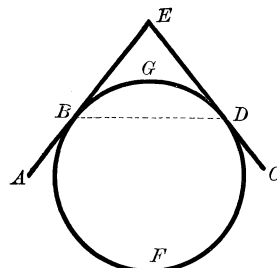


FIG. 2.

Draw figures in accordance with the statement, AE tangent to $\odot BDF$ at B , and CE a chord in Fig. 1, a secant in Fig. 2. We then have:

1. **Given** AE a tangent to $\odot BDC$ at B , and EC a secant intersecting the circumference at C and D . (Fig. 1.)

To Prove that $\angle E$ is measured by $\frac{1}{2}(\text{arc } BFC - \text{arc } BD)$.
(We have $\angle E = \angle ABC - \angle C$.)

2. (In Fig. 2, $\angle E = \angle ABD - \angle BDE$; then use § 197.)

Ex. 28. In Ex. 27 find the point of intersection of these chords when A and B are equal.

Ex. 29. Two chords intersecting in a circle intercept opposite arcs on the circumference 58° and 14° , respectively. Find the angle formed by the chords.

Ex. 30. Two unequal chords intersect at right angles. Find the sum of the opposite intercepted arcs. Will this sum remain constant if the lengths of the chords vary?

Ex. 31. A chord subtends an arc of 144° . Through one extremity of the chord a tangent is drawn; find the angles which it forms with the chord.

Ex. 32. The angle formed by a tangent and a secant is 94° , and one of the intercepted arcs is 210° ; find the other.

Ex. 33. Find the angle formed by two secants intersecting without a circle, if the intercepted arcs are 36° and 144° , respectively; also if the angle is 75° and one of the intercepted arcs is 44° , find the other.

Ex. 34. An inscribed angle is 32° . How many degrees in the arc in which it is inscribed?

Ex. 35. In how large a circle can a triangle whose sides are 12, 13, and 5 be inscribed? What kind of a triangle is it?

Ex. 36. The angle formed by two tangents is 76° ; find the intercepted arcs; also for 92° .

Ex. 37. An inscribed quadrilateral subtends arcs of 70° , 80° , 100° , and 110° , respectively. Find the angles of the quadrilateral, and also the angles of the quadrilateral formed by the intersections of the tangents through the vertices of the quadrilateral.

Ex. 38. The sum of the degrees in the angle formed by two tangents and the smaller of the intercepted arcs is 180° .

Ex. 39. Choose any two of the following statements for the hypothesis and any one of the remaining three for the conclusion, and prove each of the ten theorems thus formed, not already proved in §§ 162, 163 :

A straight line that

1. Passes through the centre of a circle ;
2. Bisects the chord ;
3. Is perpendicular to the chord ;
4. Bisects the smaller arc ;
5. Bisects the greater arc.

Ex. 40. Give a general rule for the measurement of angles formed by two lines intersecting

- (a) within a circumference,
- (b) on a circumference,
- (c) without a circumference.

Ex. 41. What figure is formed by the bisectors of the angles of a parallelogram? Give proof.

Ex. 42. Two equal chords intersect either within a circumference, on the circumference, or without the circumference. Prove that in either case the bisector of one of the angles formed by the chords passes through the centre of the circle.

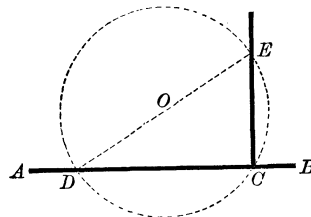
Ex. 43. Given a chord CD and A and B , two points on the circumference, such that line $AC = AD$ and line $BC = BD$, prove that line AB is a diameter.

CONSTRUCTIONS

PROP. XXVI. PROBLEM

201. *At a given point in a straight line to erect a perpendicular to that line.*

In § 24, we gave a method for drawing a perpendicular to a straight line at a given point; the following method is advantageous when the point is near the end of the line.



Given C any point in line AB .

Required to draw a line $\perp AB$ at C .

Construction. 1. With any point O without line AB as a centre, and distance OC as a radius, describe a circumference intersecting AB at C and D .

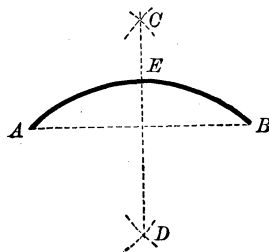
2. Draw diameter DE , and line CE .

3. Then, CE is $\perp AB$ at C .

Proof. 4. $\angle DCE$, being inscribed in a semicircle, is a rt. $\angle$. (§ 194)

PROP. XXVII. PROBLEM

202. To bisect a given arc.



Given arc AB .

Required to bisect arc AB .

Construction. 1. With A and B as centres, and with equal radii, describe arcs intersecting at C and D .

2. Draw line CD intersecting arc AB at E .

3. Then E is the middle point of arc AB .

Proof. 4. Draw chord AB .

5. Then, CD is $\perp$ chord AB at its middle point. (?)

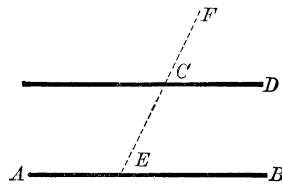
6. Whence, CD bisects arc AB . (§ 163)

Ex. 44. The hypotenuse of an isosceles right triangle is 18. Construct the triangle.

Ex. 45. Given two angles of a triangle, to construct the third. (Compare § 84.)

PROP. XXVIII. PROBLEM

203. *Through a given point without a given straight line, to draw a parallel to the line.*



Given C any point without line AB .

Required to draw through C a line $\parallel AB$.

Construction. Through C draw any line EF , meeting AB at E , and construct $\angle FCD = \angle CEB$ (§ 28); then, $CD \parallel AB$.
(The proof is left to the pupil.)

PROP. XXIX. PROBLEM

204. *Given two sides of a triangle, and the angle opposite to one of them, to construct the triangle.*

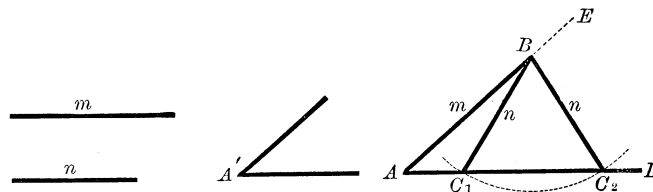
Given m and n sides of a Δ , and A' the $\angle$ opposite to n .

Required to construct the Δ .

Construction. 1. Construct $\angle DAE = \angle A'$ (§ 28); on AE take $AB = m$.

2. With B as a centre and n as a radius, describe an arc.

Case I. When A' is acute, and $m > n$.



There may be three cases :

- I. The arc may intersect AD in two points.
3. Let C_1 and C_2 be the points in which the arc intersects AD , and draw lines BC_1 and BC_2 .
4. Then, either ABC_1 or ABC_2 is the required $\triangle$.

This is called the *ambiguous case*.

- II. The arc may be tangent to AD .

In this case there is but one $\triangle$.

5. Since a tangent to a $\odot$ is $\perp$ the radius drawn to the point of contact (§ 170), the $\triangle$ is a *right* $\triangle$.

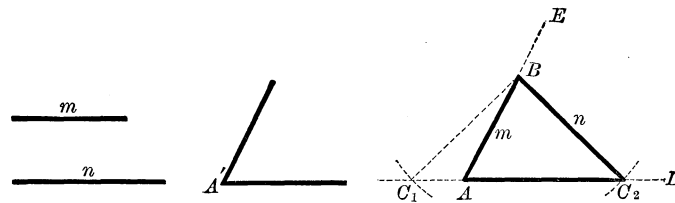
- III. The arc may not intersect AD at all.

In this case the problem is impossible.

Case II. When A' is acute, and $m = n$.

6. In this case, the arc intersects AD in two points, of which A is one; there is but one $\triangle$, an isosceles $\triangle$.

Case III. When A' is acute, and $m < n$.



7. In this case, the arc intersects AD in two points.
8. Let C_1 and C_2 be the points in which the arc intersects AD , and draw lines BC_1 and BC_2 .
9. Now $\triangle ABC_1$ does not satisfy the conditions of the problem, since it does not contain $\angle A'$.
10. Then there is but one $\triangle$; $\triangle ABC_2$.

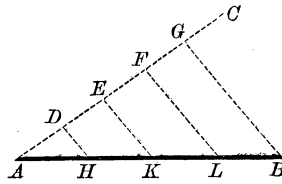
Case IV. When A' is right or obtuse, and $m < n$.

11. In each of these cases, the arc intersects AD in two points on opposite sides of A , and there is but one $\triangle$.

The pupil should construct the triangle corresponding to each case of § 204.

PROP. XXX. PROBLEM

205. *To divide a given straight line into any number of equal parts.*



Given line AB .

Required to divide AB into four equal parts.

Construction. 1. On the indefinite line AC , take any convenient length AD .

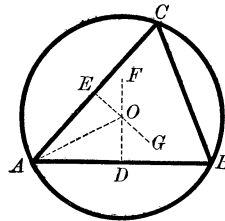
2. On DC take $DE = AD$; on EC take $EF = AD$; on FC take $FG = AD$; and draw line BG .

3. Draw lines DH , EK , and $FL \parallel BG$, meeting AB at H , K , and L , respectively.

4. Then, $AH = HK = KL = LB$. (§ 128)

PROP. XXXI. PROBLEM

206. *To circumscribe a circle about a given triangle.*



Given $\triangle ABC$.

Required to circumscribe a $\odot$ about $\triangle ABC$.

Construction. 1. Draw lines DF and $EG \perp AB$ and AC , respectively, at their middle points, intersecting at O . (§ 26)

2. With O as a centre, and OA as a radius, describe a $\odot$.
3. The circumference will pass through A , B , and C .

(The proof is left to the pupil; see § 136.)

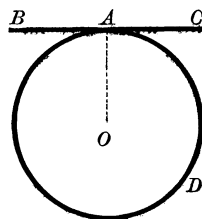
Note. The above construction serves to describe a circumference through three given points not in the same straight line, or to find the centre of a given circumference or arc.

Ex. 46. Through any given point within a circumference to draw a chord of given length.

Ex. 47. To inscribe a circle in a given triangle. (Compare § 134.)

PROP. XXXII. PROBLEM

207. *To draw a tangent to a circle through a given point on the circumference.*



Given A any point on the circumference of $\odot AD$.

Required to draw through A a tangent to $\odot AD$.

- Construction.**
1. Draw radius OA .
 2. Through A draw line $BC \perp OA$. (§ 24)
 3. Then, BC will be tangent to $\odot AD$. (?)

Ex. 48. Given two sides and the included angle of a parallelogram, to construct the parallelogram.

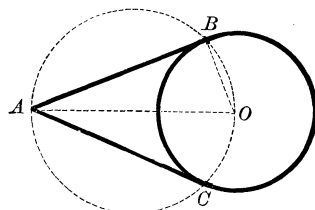
Ex. 49. Circumscribe a parallelogram about a circle and find relations between the lengths of the adjacent sides.

Ex. 50. To draw a tangent to a given circle which shall be perpendicular to a given line.

Ex. 51. To draw the shortest chord through a given point within a given circumference.

PROP. XXXIII. PROBLEM

208. To draw a tangent to a circle through a given point without the circle.



Given A any point without $\odot BC$.

Required to draw through A a tangent to $\odot BC$.

Construction. 1. Let O be the centre of $\odot BC$.

2. Draw line OA , and with OA as a diameter, describe a circumference, cutting the given circumference at B and C .

3. Draw lines AB and AC .

4. Then, AB and AC are tangents to $\odot BC$.

Proof. 5. Draw line OB .

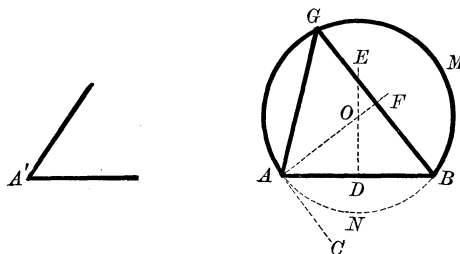
6. Since $\angle ABO$ is inscribed in a semicircle, it is a rt. $\angle$. (?)

7. Then, AB is tangent to $\odot BC$. (?)

In like manner, we may prove AC tangent to $\odot BC$.

PROP. XXXIV. PROBLEM

209. Upon a given straight line, to describe a segment which shall contain a given angle.



Given line AB , and $\angle A'$.

Required to describe upon AB a segment such that every $\angle$ inscribed in the segment shall equal $\angle A'$.

Construction. 1. Construct $\angle BAC = \angle A'$. (§ 28)
 2. Draw line $DE \perp AB$ at its middle point. (§ 26)
 3. Draw line $AF \perp AC$, intersecting DE at O .
 4. With O as centre and OA as radius, describe $\odot AMBN$.
 5. Then, AMB will be the required segment.

Proof. 6. If AGB be any $\angle$ inscribed in segment AMB , it is measured by $\frac{1}{2}$ arc ANB . (?)

7. But, by cons., $AC \perp OA$; and AC is tangent to $\odot AMB$. (?)

8. Therefore, $\angle BAC$ is measured by $\frac{1}{2}$ arc ANB . (§ 196)

9. Then, $\angle AGB = \angle BAC = \angle A'$. (?)

10. Hence, every $\angle$ inscribed in segment AMB equals $\angle A'$. (§ 193)

Ex. 52. From a given point without a circle to draw two equal secants terminating in the circumference. Can more than two be drawn?

Ex. 53. Trisect a right angle.

(In general, an angle cannot be trisected by methods of Euclidean geometry.)

Ex. 54. In a circle whose radius is 12, draw a chord whose length is 8, parallel to a given line.

Ex. 55. One of the equal sides of an isosceles triangle is 8, and one of the equal angles 30° . Construct the triangle.

Ex. 56. The distance between two parallel lines is a , and a line is drawn intersecting these two parallels. Construct a circle which shall be tangent to the three lines. How many such circles are possible?

Ex. 57. The sides of a triangle are a , b , and c , respectively. The angle opposite b is twice, and the angle opposite c three times, the angle opposite a . Construct the triangle.

Ex. 58. The base of an isosceles triangle is 10, and the perimeter 18. Construct the triangle.

Ex. 59. In a rhombus $ABCD$, $\angle A$ is twice $\angle B$. The shorter diagonal is 9. Construct the rhombus.

Ex. 60. Two angles of an inscribed quadrilateral are 90° and 60° , respectively. Find the remaining angles and inscribe the quadrilateral in a given circle. Can more than one such quadrilateral be inscribed?

Ex. 61. One side of a triangle is three-fourths another, and three-fifths the third side. The perimeter of the triangle is 48. Construct the triangle.

Ex. 62. A given point lies within, without, or on the circumference of, a circle. With the given point as centre, describe a circumference passing through the extremities of a given diameter of the circle.

Ex. 63. We define the angle between two intersecting curves as the angle formed by tangents to the curves at their point of intersection. With a point outside a given circle as a centre, describe an arc which shall intersect the given circumference at right angles.

BOOK III

PROPORTION.—SIMILAR POLYGONS

DEFINITIONS

210. A *Proportion* is a statement that two ratios are equal.

211. The statement that the ratio of a to b is equal to the ratio of c to d is written

$$\frac{a}{b} = \frac{c}{d}.$$

212. In the proportion $\frac{a}{b} = \frac{c}{d}$, we call a the *first* term, b the *second* term, c the *third* term, and d the *fourth* term.

213. We call the first and fourth terms the *extremes*, and the second and third terms the *means*.

We call the first and third terms the *antecedents*, and the second and fourth terms the *consequents*.

Thus, in the above proportion a and d are the extremes, b and c the means, a and c the antecedents, and b and d the consequents.

214. If the means of a proportion are equal, either mean is called a *mean proportional* between the first and last terms, and the last term a *third proportional* to the first and second terms.

Thus, in the proportion $\frac{a}{b} = \frac{b}{c}$, b is a mean proportional between a and c , and c a third proportional to a and b .

215. In the proportion $\frac{a}{b} = \frac{c}{d}$, d is called a *fourth proportional* to a , b , and c .

PROP. I. THEOREM

216. *In any proportion, the product of the extremes is equal to the product of the means.*

Given the proportion $\frac{a}{b} = \frac{c}{d}$. (1)

To Prove $ad = bc$.

Proof. Multiplying both members of equation (1) by bd ,

$$ad = bc.$$

217. Applying the above theorem to the proportion

$$\frac{a}{b} = \frac{b}{c}, \text{ we have } b^2 = ac, \text{ or } b = \sqrt{ac}.$$

That is, *the mean proportional between two numbers is equal to the square root of their product.*

PROP. II. THEOREM

218. (Converse of Prop. I.) *If the product of two numbers is equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.*

Given $ad = bc$. (1)

To Prove $\frac{a}{b} = \frac{c}{d}$.

Proof. Dividing both members of (1) by bd ,

$$\frac{ad}{bd} = \frac{bc}{bd}, \text{ or } \frac{a}{b} = \frac{c}{d}.$$

In like manner, we may prove $\frac{a}{c} = \frac{b}{d}$; $\frac{b}{a} = \frac{d}{c}$; etc.

PROP. III. THEOREM

219. *In any proportion, the terms are in proportion by ALTERNATION; that is, the first term is to the third as the second term is to the fourth.*

Given the proportion $\frac{a}{b} = \frac{c}{d}$. (1)

To Prove $\frac{a}{c} = \frac{b}{d}.$

Proof. From (1), $ad = bc.$ (§ 216)

Then, $\frac{a}{c} = \frac{b}{d}.$ (§ 218)

PROP. IV. THEOREM

220. *In any proportion, the terms are in proportion by INVERSION; that is, the second term is to the first as the fourth term is to the third.*

Given the proportion $\frac{a}{b} = \frac{c}{d}.$ (1)

To Prove $\frac{b}{a} = \frac{d}{c}.$

Proof. From (1), $ad = bc.$ (?)

Then, $\frac{b}{a} = \frac{d}{c}.$ (?)

PROP. V. THEOREM

221. *In any proportion, the terms are in proportion by COMPOSITION; that is, the sum of the first two terms is to the first term as the sum of the last two terms is to the third term.*

Given the proportion $\frac{a}{b} = \frac{c}{d}.$ (1)

To Prove $\frac{a+b}{a} = \frac{c+d}{c}.$

Proof. From (1), $ad = bc.$ (?)

Adding both members of this equation to ac ,

$$ac + ad = ac + bc, \text{ or } a(c + d) = c(a + b).$$

Then, $\frac{a+b}{a} = \frac{c+d}{c}$ (§ 218). (2)

In like manner, we may prove $\frac{a+b}{b} = \frac{c+d}{d}.$

PROP. VI. THEOREM

222. *In any proportion, the terms are in proportion by DIVISION; that is, the difference of the first two terms is to the first term as the difference of the last two terms is to the third term.*

Given the proportion $\frac{a}{b} = \frac{c}{d}$. (1)

To Prove $\frac{a-b}{a} = \frac{c-d}{c}$.

Proof. From (1), $ad = bc$. (?)

Subtracting both members from ac ,

$$ac - ad = ac - bc, \text{ or } a(c-d) = c(a-b).$$

Then, $\frac{a-b}{a} = \frac{c-d}{c}$ (§ 218). (2)

In like manner, we prove $\frac{a-b}{b} = \frac{c-d}{d}$.

PROP. VII. THEOREM

223. *In any proportion, the terms are in proportion by COMPOSITION AND DIVISION; that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Given the proportion $\frac{a}{b} = \frac{c}{d}$.

To Prove $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

Proof. Dividing equation (2), § 221, by equation (2), § 222,

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

Ex. 1. What is the ratio of 50 cents to $\$1\frac{1}{2}$? of $\frac{1}{2}$ hour to 1 day? of 150 rods to 1 mile?

Ex. 2. If x is a fourth proportional to a , b , c , find x .

Ex. 3. Apply composition and division to the following:

$$\frac{x + \sqrt{x^2 - a^2}}{x - \sqrt{x^2 - a^2}} = \frac{2a + b}{2a - b}.$$

PROP. VIII. THEOREM

224. *In a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

Given $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}. \quad (1)$

To Prove $\frac{a + c + e}{b + d + f} = \frac{a}{b}.$

Proof. We have $ba = ab.$

And from (1), $bc = ad,$

and $be = af. \quad (?)$

Adding, $ba + bc + be = ab + ad + af.$

Or, $b(a + c + e) = a(b + d + f).$

Then, $\frac{a + c + e}{b + d + f} = \frac{a}{b}. \quad (?)$

225. The ratio of two magnitudes of the same kind is equal to the ratio of their numerical measures when referred to a common unit (§ 183).

Then, in every proportion involving the ratio of magnitudes of the same kind, we shall regard the ratio of the magnitudes as replaced by *the ratio of their numerical measures when referred to a common unit.*

Thus, if AB and CD are lines, and

$$\frac{AB}{CD} = \frac{m}{n},$$

where m and n are numbers, this means simply that the *ratio of the numerical measures of AB and CD , when referred to a common unit, is equal to the ratio of m to n .*

Ex. 4. Find a third proportional to $\frac{1}{3}$ and $\frac{7}{12}$.

Ex. 5. Find a mean proportional between $\frac{25}{14}$ and $\frac{224}{9}$.

Ex. 6. If the ratio of the hypotenuse of a right triangle to one of the legs is 2, the acute angles are 30° and 60° .

Ex. 7. Each of two angles of a triangle is 45° . Find the ratio of the sides opposite these angles.

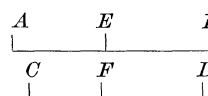
Ex. 8. Two angles of a triangle are 30° and 60° , respectively. Find the ratio of the sides opposite these angles. Is the ratio commensurable?

PROPORTIONAL LINES

226. Def. Two straight lines are said to be *divided proportionally* when their corresponding segments are in the same ratio as the lines themselves.

Thus, lines AB and CD are divided proportionally if

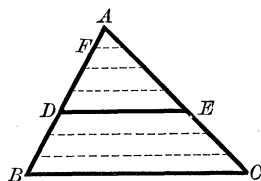
$$\frac{AE}{CF} = \frac{BE}{DF} = \frac{AB}{CD}.$$



PROP. IX. THEOREM

227. A parallel to one side of a triangle divides the other two sides proportionally.

Case I. When the segments of each side are commensurable.



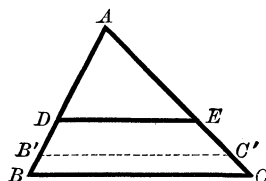
Draw $\triangle ABC$. Take D , a point in AB , such that AD and DB are commensurable; draw $DE \parallel BC$, meeting AC at E . We then have:

Given, in $\triangle ABC$, segments AD and BD of side AB commensurable, and line $DE \parallel BC$, meeting AC at E .

To Prove $\frac{AD}{BD} = \frac{AE}{CE}.$

- Proof.** 1. Let AF be contained 4 times in AD , 3 times in BD .
 2. Find ratio of AD to BD .
 3. Through points of division of AB draw $\parallel$ s to BC .
 4. Find ratio of AE to CE . (§ 128)
 5. From steps (2) and (4), $\frac{AD}{BD} = \frac{AE}{CE}.$ (?)

Case II. *When the segments of each side are incommensurable.*



Draw figure in accordance with the statement of the proposition. We then have :

Given, in $\triangle ABC$, segments AD and BD of side AB incommensurable, and line $DE \parallel BC$, meeting AC at E .

To Prove
$$\frac{AD}{BD} = \frac{AE}{CE}.$$

Proof. 1. Let AD be divided into any number of equal parts, and let one of these parts be applied to BD as a unit of measure.

2. Since AD and BD are incommensurable, a certain number of the equal parts will extend from D to B' , leaving a remainder $BB' <$ one of the equal parts.

3. Draw line $B'C' \parallel BC$, meeting AC at C' .

4. Then, AD and $B'D$ are commensurable, and

$$\frac{AD}{B'D} = \frac{AE}{C'E}. \quad (\text{Case I})$$

5. Now, if the number of subdivisions of AD be indefinitely increased, the unit of measure will be indefinitely diminished, and the remainder BB' will approach the limit 0.

6. Then, $\frac{AD}{B'D}$ will approach the limit $\frac{AD}{BD}$,

and $\frac{AE}{C'E}$ will approach the limit $\frac{AE}{CE}$.

7. By the Theorem of Limits, these limits are equal. (?)

8. Then,
$$\frac{AD}{BD} = \frac{AE}{CE}.$$

The above result simply means that the ratio of the numerical measures of lines AD and BD , when referred to a common unit, is equal to ratio of the numerical measures of lines AE and CE , when referred to a common unit.

A similar meaning is attached to every proportion involving the ratio of two geometrical magnitudes of the same kind. (Compare § 225.)

228. Applying the theorem of § 221 to the proportion of § 227, we have

$$\frac{AD + BD}{AD} = \frac{AE + CE}{AE}, \text{ or } \frac{AB}{AD} = \frac{AC}{AE}. \quad (1)$$

In like manner,
$$\frac{AB}{BD} = \frac{AC}{CE}. \quad (2)$$

Again, applying the theorem of § 220 to the proportions (1) and (2), we obtain the proportions

$$\frac{AB}{AC} = \frac{AD}{AE}, \text{ and } \frac{AB}{AC} = \frac{BD}{CE}.$$

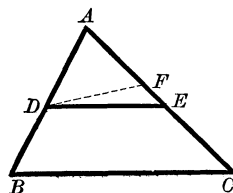
Then by Ax. 1,
$$\frac{AB}{AC} = \frac{AD}{AE} = \frac{BD}{CE}. \quad (3)$$

The proportions (1), (2), and (3) are all included in the statement of Prop. IX:

A parallel to one side of a triangle divides the other two sides proportionally.

PROP. X. THEOREM

229. (Converse of Prop. IX.) *A line which divides two sides of a triangle proportionally is parallel to the third side.*



Draw $\triangle ABC$. Draw line DE dividing AB and AC proportionally at D and E , respectively. We then have:

Given, in $\triangle ABC$, line DE meeting AB and AC at D and E , respectively, so that

$$\frac{AB}{AD} = \frac{AC}{AE}.$$

To Prove $DE \parallel BC$.

Proof. 1. If DE is not $\parallel BC$, draw line $DF \parallel BC$, meeting AC at F .

$$2. \text{ Then, } \frac{AB}{AD} = \frac{AC}{AF}. \quad (\S 228)$$

$$3. \text{ But by hyp., } \frac{AB}{AD} = \frac{AC}{AE}.$$

$$4. \text{ Then, } \frac{AC}{AE} = \frac{AC}{AF} \text{ or } AE = AF. \quad (?)$$

5. Then, line DF coincides with DE , and line DE is $\parallel BC$.
(Ax. 3)

PROP. XI. THEOREM

230. *In any triangle, the bisector of an angle divides the opposite side into segments proportional to the adjacent sides.*

Draw $\triangle ABC$. Draw line AD , bisecting $\angle A$, meeting BC at D . We then have :

Given line AD bisecting $\angle A$ of $\triangle ABC$, meeting BC at D .

To Prove $\frac{DB}{DC} = \frac{AB}{AC}$.

Proof. 1. Draw line $BE \parallel AD$, meeting CA extended at E ; represent $\angle BAD$, CAD , and ABE by a , b , c , respectively.

$$2. \text{ We have } \angle c = \angle a. \quad (?)$$

$$3. \text{ Also, } \angle E = \angle b. \quad (?)$$

$$4. \text{ By hyp., } \angle a = \angle b.$$

$$5. \text{ Then, } \angle c = \angle E. \quad (?)$$

$$6. \text{ Then, } AB = AE. \quad (?)$$

$$7. \text{ Now } \frac{DB}{DC} = \frac{AE}{AC}. \quad (\S 228)$$

8. Then,
$$\frac{DB}{DC} = \frac{AB}{AC}. \quad (?)$$

231. Def. The *segments* of a line by a point are the distances from the point to the extremities of the line, whether the point is in the line itself, or in the line extended.

Ex. 9. Three parallels cut segments 5 and 3, respectively, from a line drawn at random across them. What is the ratio of the segments cut from another line drawn across these three parallels?

Ex. 10. A line 15 inches long is parallel to the base of a triangle. It divides the sides in the ratio of 3 to 2. Find the base.

PROP. XII. THEOREM

232. *In any triangle the bisector of an exterior angle divides the opposite side externally into segments proportional to the adjacent sides.*

Draw $\triangle ABC$. Extend CA to E and draw line AD bisecting $\angle BAE$, meeting CB extended at D . We then have:

Given line AD bisecting ext. $\angle BAE$ of $\triangle ABC$, meeting CB extended at D .

To Prove
$$\frac{DB}{DC} = \frac{AB}{AC}.$$

(Draw $BF \parallel AD$; then $\angle ABF = \angle AFB$, and $AF = AB$; BF is $\parallel$ to side AD of $\triangle ACD$.)

The above theorem is not true for the exterior angle at the vertex of an isosceles triangle.

Ex. 11. The base of a triangle is 25, and the other sides are 15 and 16, respectively. Draw a line which shall be parallel to the base of the triangle, terminate in the sides, and be 21 in length.

Ex. 12. One of the parallel sides of a trapezoid is double the other. In what ratio do the diagonals intersect?

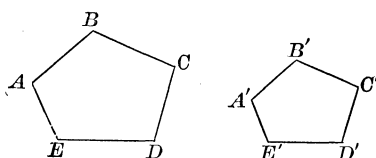
Ex. 13. In a triangle ABC , $AB = 12$, $BC = 15$, $CA = 13$. Find the segments into which the bisector of $\angle B$ divides the opposite side.

Ex. 14. The sides of a triangle are 8, 10, and 12; find the segments of the side 12 made by the bisector of the opposite angle.

Ex. 15. The sides of a triangle are $AB = 6$, $BC = 7$, $CA = 8$. Find the length of the segments into which the bisector of the exterior angle at B divides AC .

SIMILAR POLYGONS

233. Def. Two polygons are said to be *similar* if they are mutually equiangular (§ 121), and have their homologous sides proportional.



Thus, polygons $ABCDE$ and $A'B'C'D'E'$ are similar if

$$\angle A = \angle A', \angle B = \angle B', \text{ etc.,}$$

and,

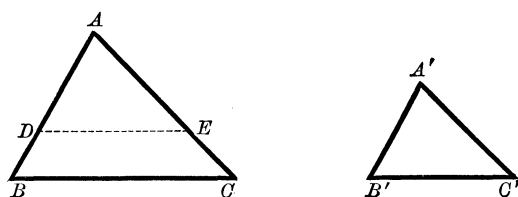
$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

234. The following are given for reference :

1. *In similar polygons, the homologous angles are equal.*
2. *In similar polygons, the homologous sides are proportional.*

PROP. XIII. THEOREM

235. *Two triangles are similar when they are mutually equiangular.*



Draw $\triangle ABC$; also, $\triangle A'B'C'$ having $\angle A' = \angle A$, $\angle B' = \angle B$, and $\angle C' = \angle C$. We then have :

Given, in $\triangle ABC$ and $A'B'C'$,

$$\angle A = \angle A', \angle B = \angle B', \text{ and } \angle C = \angle C'.$$

To Prove $\triangle ABC$ and $A'B'C'$ similar.

Proof. 1. Place $\triangle A'B'C'$ in position ADE ; $\angle A'$ coinciding with $\angle A$, vertices B' and C' falling at D and E , respectively, and side $B'C'$ at DE .

2. We have
$$\frac{AB}{AD} = \frac{AC}{AE}. \quad (\S\ 228)$$

3. Then,
$$\frac{AB}{A'B'} = \frac{AC}{A'C'}. \quad (?)$$

4. In like manner, by placing B' at B ,

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}.$$

5. Then,
$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}. \quad (?)$$

6. Then $\triangle ABC$ and $A'B'C'$ have their homologous sides proportional, and are similar. (§ 233)

The following are consequences of § 235:

236. *Two triangles are similar when two angles of one are equal respectively to two angles of the other.*

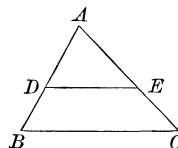
For their remaining $\angle$ s are equal each to each. (§ 87)

237. *Two right triangles are similar when an acute angle of one is equal to an acute angle of the other.*

238. *If a line be drawn between two sides of a triangle parallel to the third side, the triangle formed is similar to the given triangle.*

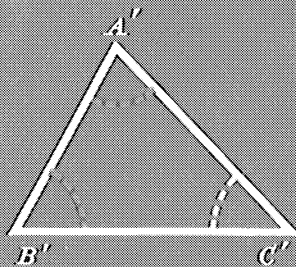
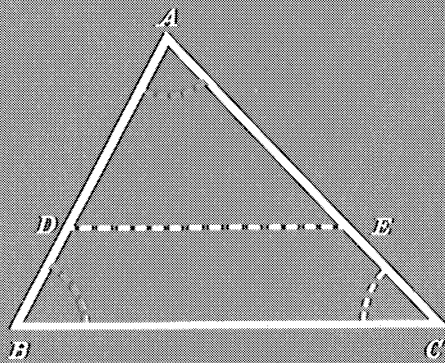
For the triangles are mutually equiangular. (§ 76)

239. Note. *In similar triangles, the homologous sides lie opposite the equal angles.*



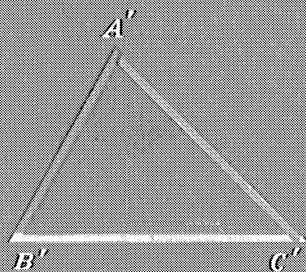
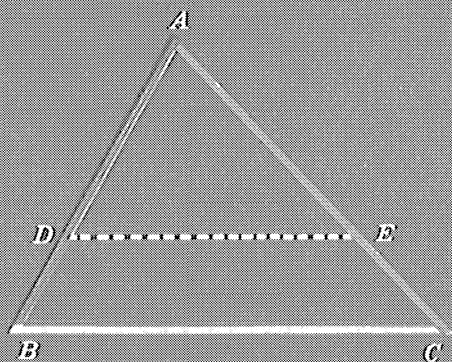
Ex. 16. Find the sides of a triangle similar to that in Ex. 14, one side being 9. Does more than one triangle satisfy this condition?

Ex. 17. Each of two isosceles triangles has a vertical angle of 32° . Do their homologous sides form a proportion? Why?



PROP. XIII.

PLATE V

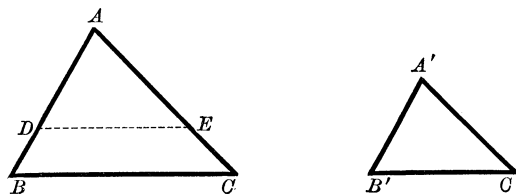


PROP. XIV.

PLATE VI

PROP. XIV. THEOREM

240. *Two triangles are similar when their homologous sides are proportional.*



Draw $\triangle ABC$ and $A'B'C'$ having their homologous sides proportional. We then have :

Given, in $\triangle ABC$ and $A'B'C'$,

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}. \quad (1)$$

To Prove $\triangle ABC$ and $A'B'C'$ similar.

Proof. 1. On AB and AC , take $AD = A'B'$ and $AE = A'C'$; and draw line DE .

2. Since $\frac{AB}{AD} = \frac{AC}{AE}$, $DE \parallel BC$. (§ 229)

3. Then $\triangle ADE$ and ABC are similar. (§ 238)

4. Then $\frac{AB}{AD} = \frac{BC}{DE}$, or $\frac{AB}{A'B'} = \frac{BC}{DE}$. (§ 234, 2)

5. Then $DE = B'C'$ by comparing this with (1).

6. Then $\triangle ADE = \triangle A'B'C'$. (?)

7. Then, $\triangle A'B'C'$ is similar to $\triangle ABC$.

Ex. 18. The sides of a triangle are 7, 12, and 14. If the side of a similar triangle homologous to 12 is 18, find the remaining sides of the second triangle.

Ex. 19. The diagonals of any trapezoid divide each other proportionally.

Ex. 20. A post a feet high casts a shadow b feet long. At the same time, a tree casts a shadow c feet long. Find the height of the tree. What theorem of proportion is involved?

241. NOTE. To prove two polygons in general similar, it must be shown that they are mutually equiangular, and have their homologous sides proportional (§ 233); but in the case of two *triangles*, each of these conditions involves the other (§§ 235, 240), so that it is only necessary to show that one of the tests of similarity is satisfied.

PROP. XV. THEOREM

242. *Two triangles are similar when they have an angle of one equal to an angle of the other, and the sides including these angles proportional.*

Draw $\triangle ABC$ and $A'B'C'$ having $\angle A = \angle A'$, and the sides including these sides proportional. We then have :

Given, in $\triangle ABC$ and $A'B'C'$, $\angle A = \angle A'$, and $\frac{AB}{A'B'} = \frac{AC}{A'C'}$.

To Prove $\triangle ABC$ and $A'B'C'$ similar.

(Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle A'$ coincides with $\angle A$ and vertices B' and C' fall at D and E , respectively; by § 229, $DE \parallel BC$; the theorem follows by § 238.)

PROP. XVI. THEOREM

243. *Two triangles are similar when their sides are parallel each to each, or perpendicular each to each.*

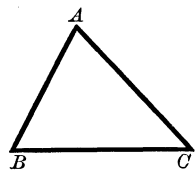


FIG. 1.

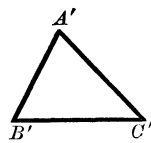


FIG. 2.

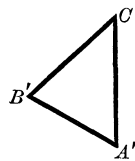


FIG. 3.

Draw $\triangle ABC$ and $A'B'C'$ in accordance with the statement of the proposition. We then have :

Given sides AB , AC , and BC , of $\triangle ABC$, $\parallel$ respectively to sides $A'B'$, $A'C'$, and $B'C'$ of $\triangle A'B'C'$ in Fig. 2, and $\perp$ respectively to sides $A'B'$, $A'C'$, and $B'C'$ of $\triangle A'B'C'$ in Fig. 3.

To Prove $\triangle ABC$ and $A'B'C'$ similar.

Proof. 1. $\angle A$ and A' are either equal or supplementary, as also are B and B' , and C and C' . (§§ 81, 82, 83)

2. We may then make the following hypotheses with regard to the $\angle$ s of the $\triangle$:

I. $A + A' = 2 \text{ rt. } \angle$, $B + B' = 2 \text{ rt. } \angle$, $C + C' = 2 \text{ rt. } \angle$.

II. $A + A' = 2 \text{ rt. } \angle$, $B + B' = 2 \text{ rt. } \angle$, $C = C'$.

III. $A + A' = 2 \text{ rt. } \angle$, $B = B'$, $C + C' = 2 \text{ rt. } \angle$.

IV. $A = A'$, $B + B' = 2 \text{ rt. } \angle$, $C + C' = 2 \text{ rt. } \angle$.

V. $A = A'$, $B = B'$, whence $C = C'$. (§ 87)

3. The first four hypotheses are impossible; for, in either case, the sum of the $\angle$ s of the two $\triangle$ would be $> 4 \text{ rt. } \angle$. (§ 84)

4. We then have only $A = A'$, $B = B'$, and $C = C'$.

5. Therefore, $\triangle ABC$ and $A'B'C'$ are similar. (§ 235)

244. Note. 1. *In similar triangles whose sides are parallel each to each, the parallel sides are homologous.*

2. *In similar triangles whose sides are perpendicular each to each, the perpendicular sides are homologous.*

Ex. 21. The bisectors of the equal angles of an isosceles triangle extended to meet the equal sides are divided proportionally.

Ex. 22. The altitudes to the equal sides of an isosceles triangle intersect in a point such that the product of segments of one equals the product of the segments of the other.

Ex. 23. The diagonals of inscribed quadrilateral $ABCD$ intersect at O . If OR and OS are the altitudes of triangles OBC and OAD , respectively, prove

$$\frac{OR}{OS} = \frac{BC}{AD}.$$

PROP. XVII. THEOREM

245. *The homologous altitudes of two similar triangles are in the same ratio as any two homologous sides.*

Draw $\triangle ABC$, and line $AD \perp BC$. Draw $\triangle A'B'C'$, making $\angle B'A'C' = \angle BAC$ and $\angle B' = \angle B$; draw line $A'D' \perp B'C'$. We now have:

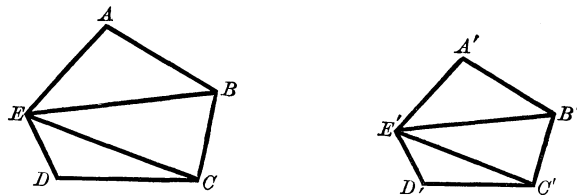
Given AD and $A'D'$ homologous altitudes of similar $\triangle ABC$ and $A'B'C'$.

To Prove $\frac{AD}{A'D'} = \frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$

(Rt. $\triangle ABD$ and $A'B'D'$ are similar by § 237.)

PROP. XVIII. THEOREM

246. *Two polygons are similar when they are composed of the same number of triangles, similar each to each, and similarly placed.*



Draw polygon $ABCDE$; from E draw diagonals EC and EB . Construct $\triangle E'D'C'$ similar to $\triangle EDC$, sides DC and $D'C'$ being homologous; on $E'C'$, homologous to EC , construct $\triangle E'C'B'$ similar to $\triangle ECB$; on $E'B'$, homologous to EB , construct $\triangle E'B'A'$ similar to $\triangle EBA$. We then have:

Given, in polygons AC and $A'C'$, $\triangle ABE$ similar to $\triangle A'B'E'$, $\triangle BCE$ to $\triangle B'C'E'$, and $\triangle CDE$ to $\triangle C'D'E'$.

To Prove polygons AC and $A'C'$ are similar.

Proof. 1. $\triangle ABE$ and $A'B'E'$ are similar; find equal parts of these $\triangle$.

2. $\triangle BCE$ and $B'C'E'$ are similar; find equal parts of these $\triangle$.

3. Now, $\angle ABE + \angle EBC = \angle A'B'E' + \angle E'B'C'$, or
 $\angle ABC = \angle A'B'C'.$

4. In like manner, $\angle BCD = \angle B'C'D'$, etc.; and AC and $A'C'$ are mutually equiangular.

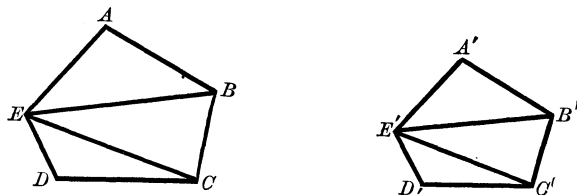
5. Then $\frac{AB}{A'B'} = \frac{BE}{B'E'}$ and $\frac{BE}{B'E'} = \frac{BC}{B'C'}$; whence $\frac{AB}{A'B'} = \frac{BC}{B'C'}$

6. In like manner, $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$, etc.

7. Then AC and $A'C'$ are similar. (§ 233)

PROP. XIX. THEOREM

247. (Converse of Prop. XVIII.) *Two similar polygons may be decomposed into the same number of triangles, similar each to each, and similarly placed.*



Draw similar polygons $ABCDE$, $A'B'C'D'E'$, vertices E , E' being homologous; and diagonals EB , EC , $E'B'$, $E'C'$. We then have:

Given E and E' homologous vertices of similar polygons AC and $A'C'$, and lines EB , EC , $E'B'$, and $E'C'$.

To Prove $\triangle ABE$ similar to $\triangle A'B'E'$, $\triangle BCE$ to $\triangle B'C'E'$, and $\triangle CDE$ to $\triangle C'D'E'$.

Proof. 1. In similar polygons, the homologous $\angle$ s are equal, and the homologous sides proportional.

2. Then $\triangle ABE$ and $\triangle A'B'E'$ are similar. (§ 242)

3. Using other $\angle$ s and sides of AC and $A'C'$, and of $\triangle ABE$ and $\triangle A'B'E'$, prove $\triangle BCE$ and $\triangle B'C'E'$ similar.

4. In like manner, prove $\triangle CDE$ and $\triangle C'D'E'$ similar.

PROP. XX. THEOREM

248. *The perimeters of two similar polygons are in the same ratio as any two homologous sides.*

Draw similar polygons $ABCDE$, $A'B'C'D'E'$, vertices A , A' being homologous. We then have:

Given AB and $A'B'$, BC and $B'C'$, CD and $C'D'$, etc., homologous sides of similar polygons AC and $A'C'$.

To Prove

$$\frac{AB + BC + CD + \text{etc.}}{A'B' + B'C' + C'D' + \text{etc.}} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

(Apply § 224 to the equal ratios of § 233.)

Ex. 24. From any point in the base of an isosceles triangle perpendiculars to the equal sides are drawn. Prove that these perpendiculars form equal angles with the base and that they are proportional to the segments they cut off from the equal sides; the segments being measured from the extremities of the base towards the vertex.

Ex. 25. A straight line bisecting the exterior angle at the vertex of a triangle is parallel to the base; find the ratio of the sides of the triangle.

PROP. XXI. THEOREM

249. *If a perpendicular be drawn from the vertex of the right angle to the hypotenuse of a right triangle,*

I. *The triangles formed are similar to the whole triangle, and to each other.*

II. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

III. *Either leg is a mean proportional between the whole hypotenuse and the adjacent segment.*

Draw $\triangle ABC$, with rt. $\angle$ at C ; also, line $CD \perp AB$. We then have:

Given line $CD \perp$ hypotenuse AB of rt. $\triangle ABC$.

To Prove $\triangle ACD$ and BCD similar to $\triangle ABC$, and to each other; also,

$$\frac{AD}{CD} = \frac{CD}{BD}, \frac{AB}{AC} = \frac{AC}{AD}, \text{ and } \frac{AB}{BC} = \frac{BC}{BD}.$$

Proof. 1. Rt. $\triangle ACD$ and ABC are similar. (§ 237)

2. In like manner, $\triangle BCD$ and ABC are similar.

3. Then, $\triangle ACD$ and BCD are similar.

4. In similar $\triangle$ s, homologous sides lie opposite equal $\angle$ s (§ 239); find sides of $\triangle BCD$ homologous to sides AD and CD of $\triangle ACD$.

5. Then,
$$\frac{AD}{CD} = \frac{CD}{BD}. \quad (\S 234, 2)$$

6. In like manner, from $\triangle ABC$ and ACD , and $\triangle ABC$ and BCD ,

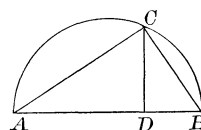
$$\frac{AB}{AC} = \frac{AC}{AD}, \text{ and } \frac{AB}{BC} = \frac{BC}{BD}.$$

250. Since an angle inscribed in a semicircle is a right angle (§ 194), it follows that:

If a perpendicular be drawn from any point in the circumference of a circle to a diameter,

1. *The perpendicular is a mean proportional between the segments of the diameter.*

2. *The chord joining the point to either extremity of the diameter is a mean proportional between the whole diameter and the adjacent segment.*



251. We have from the three proportions of § 249,

$$\overline{CD}^2 = AD \times BD,$$

$$\overline{AC}^2 = AB \times AD,$$

and

$$\overline{BC}^2 = AB \times BD. \quad (?)$$

Hence, *if a perpendicular be drawn from the vertex of the right angle to the hypotenuse of a right triangle,*

1. *The square of the perpendicular is equal to the product of the segments of the hypotenuse.*

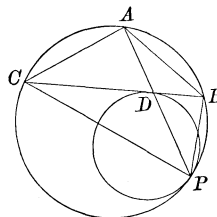
2. *The square of either leg is equal to the product of the whole hypotenuse and the adjacent segment.*

These equations mean that the square of the *numerical measure* of CD equals the product of the *numerical measures* of AD and BD ; etc. (Compare § 225.)

Ex. 26. The non-parallel sides AD and BC of trapezoid $ABCD$ intersect at O . If $AB = 15$, $CD = 24$, and the altitude of the trapezoid is 8, what is the altitude of triangle OAB ?

Ex. 27. Given two similar triangles, and the altitude of the first double the homologous altitude of the second. Find the ratio between the radii of the circles circumscribing these triangles.

Ex. 28. If two circles are tangent internally, and a chord of the greater is a tangent to the smaller, and if through the points of contact a chord of the greater circle be drawn, then the chords joining the extremities of the tangent and the chord form two pairs of similar triangles.



Ex. 29. The perimeters of two similar polygons are A and B feet, respectively. If a side of the first is a feet, find the homologous side of the second.

Ex. 30. The perimeters of two similar polygons are in the ratio of 4 to 5. If the perimeter of the second is 200, what is the perimeter of the first? If the ratio is A to B and a side of the second is b , what is the homologous side of the first?

Ex. 31. Give a method for stretching strings which mark out a rectangular building for the excavators.

Ex. 32. The legs of a right triangle are 12 and 16. Find the length of the perpendicular from the vertex of the right angle to the hypotenuse.

Ex. 33. In any right triangle if a perpendicular be drawn from the vertex of the right angle to the hypotenuse, the hypotenuse is to either leg as the other leg is to the perpendicular to the hypotenuse.

PROP. XXII. THEOREM

252. *In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.*

Draw $\triangle ABC$ with rt. $\angle$ at C . We then have :

Given $\triangle ABC$, with rt. $\angle$ at C .

To Prove $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

Proof. 1. Draw line $CD \perp AB$,

2. Then, $\overline{AC}^2 = AB \times AD$,

and $\overline{BC}^2 = AB \times BD$. (§ 251, 2)

3. Adding, $\overline{AC}^2 + \overline{BC}^2 = AB \times (AD + BD) = AB \times AB$.

4. Then, $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

253. It follows from § 252 that

$$\overline{AC}^2 = \overline{AB}^2 - \overline{BC}^2, \text{ and } \overline{BC}^2 = \overline{AB}^2 - \overline{AC}^2.$$

That is, *in any right triangle, the square of either leg is equal to the square of the hypotenuse, minus the square of the other leg.*

Ex. 34. If D is the middle point of side BC of right triangle ABC , and DE be drawn perpendicular to the hypotenuse AB , prove

$$\overline{AB}^2 - \overline{CB}^2 = \overline{AE}^2 - \overline{EB}^2.$$

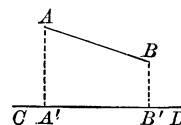
Ex. 35. One leg of a right triangle is double the other. Find the ratio of the segments of the hypotenuse formed by a perpendicular drawn from the vertex of the right angle to the hypotenuse.

Ex. 36. In an isosceles triangle ABC , the altitude CE is extended to D , so that $\angle DAE = 30^\circ$. It is then found that AE is a mean proportional between CE and ED . Find AB in terms of AC .

Ex. 37. If from the sum of the squares of the diagonals of a square we subtract the sum of the squares of the four sides of the square, the result is zero.

254. Defs. The *projection of a point* upon a straight line of indefinite length, is the foot of the perpendicular from the point to the line.

Thus, if line AA' be perpendicular to line CD , the projection of point A on line CD is point A' .



The *projection of a finite straight line* upon a straight line of indefinite length, is that portion of the second line included between the projections of the extremities of the first.

Thus, if lines AA' and BB' be perpendicular to line CD , the projection of line AB upon line CD is line $A'B'$.

PROP. XXIII. THEOREM

255. In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other side upon it.

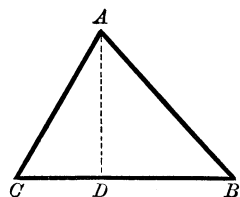


FIG. 1.

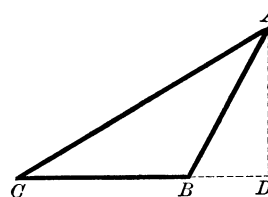


FIG. 2.

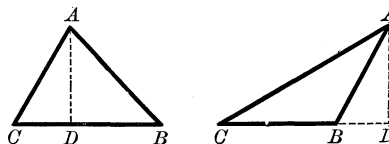
Draw acute-angled $\triangle ABC$; draw also $\triangle ABC$ having an obtuse angle at B . Let CD be projection of CA upon CB . We then have:

Given C an acute $\angle$ of $\triangle ABC$, and CD the projection of side AC upon side CB , extended if necessary. (§ 254)

To Prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2 BC \times CD.$

Proof. 1. Draw line AD ; then, $AD \perp CD.$ (§ 254)

2. There will be two cases according as D falls on CB (Fig. 1), or on CB extended (Fig. 2); in each figure express BD in terms of BC and $CD.$



3. Square both members of each of these equations.

4. Add $\overline{AD}^2$ to both members of the result.

5. Find values of $\overline{AD}^2 + \overline{BD}^2$ and $\overline{AD}^2 + \overline{CD}^2$ from the figures. (§ 252)

6. Then, $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2 BC \times CD.$

PROP. XXIV. THEOREM

256. *In any triangle having an obtuse angle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other side upon it.*

Draw $\triangle ABC$ having an obtuse angle at C ; draw $AD \perp BC$, meeting BC extended at D . We then have:

Given C an obtuse $\angle$ of $\triangle ABC$, and CD the projection of side AC upon side BC extended.

To Prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2 BC \times CD.$

(We have $BD = BC + CD$; square both members, using the algebraic rule for the square of the sum of two numbers, and then add $\overline{AD}^2$ to both members.)

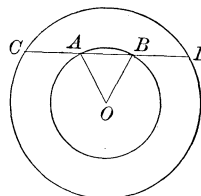
Ex. 38. In any triangle, if a perpendicular be drawn from the vertex to the base, the sum of the other sides of the triangle is to the sum of the segments of the base as the difference of the segments of the base is to the difference of the sides.

Ex. 39. In right triangle ABC , $\overline{BC}^2 = 3 \overline{AC}^2$. If line CD be drawn from the vertex of the right angle to the middle point of AB , prove $\angle ACD$ equal to 60° .

Ex. 40. If in triangle ABC , $\angle C = 120^\circ$, prove

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + AC \times BC.$$

Ex. 41. CD is a chord of the greater of two concentric circles, intersecting the smaller circumference at A and B . CD is 40, $\angle AOB$ is 60° , and AB equals BD ; find distance CO and the distance from O to CD .

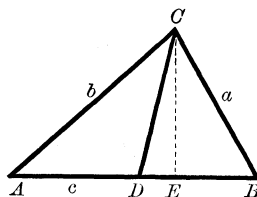


PROP. XXV. THEOREM

257. In any triangle, if a median be drawn from the vertex to the base,

I. The sum of the squares of the other two sides is equal to twice the square of half the base, plus twice the square of the median.

II. The difference of the squares of the other two sides is equal to twice the product of the base and the projection of the median upon the base.



Draw $\triangle ABC$ having $\angle B > \angle A$, and CD a median meeting AB at D . Draw line $CE \perp AB$. We then have:

Given DE the projection of median CD upon base AB (c) of $\triangle ABC$; and AC (b) $>$ BC (a).

To Prove $b^2 + a^2 = 2 \overline{AD}^2 + 2 \overline{CD}^2$, (1)

and $b^2 - a^2 = 2 c \times DE$. (2)

Proof. 1. Since $b > a$, E lies between B and D ; and $\angle ADC$ is obtuse, and $\angle BDC$ acute.

2. In $\triangle ADC$, $b^2 = \overline{AD}^2 + \overline{CD}^2 + 2 AD \times DE$ (§ 256)

$= \overline{AD}^2 + \overline{CD}^2 + c \times DE$. (3)

3. In $\triangle BDC$, $a^2 = \overline{BD}^2 + \overline{CD}^2 - 2 BD \times DE$ (§ 255)

$= \overline{AD}^2 + \overline{CD}^2 - c \times DE$. (4)

4. Adding and subtracting (3) and (4) gives (1) and (2).

Ex. 42. One side of an inscribed quadrilateral is 14. The diameter of the circle is 24. Find the distance of this side from the centre of the circle correct to three decimal places.

Ex. 43. The radii of two concentric circles are a and b , respectively. Find the length of a chord of the greater which is tangent to the smaller.

Ex. 44. Two unequal circles are tangent externally at A . BC and DE are secants drawn through A , and terminating in one circumference at B and D , in the other at C and E . Chords BD and CE are drawn, forming triangles ABD , ACE . Prove $AB \times AE = AC \times AD$.

Ex. 45. If a median of a triangle is perpendicular to the side, the triangle is isosceles.

Ex. 46. One of the equal sides of an isosceles right triangle is 12 inches in length; find the length of the median drawn to the hypotenuse.

Ex. 47. Two parallel chords are 6 inches apart. The length of the smaller chord is 14 inches, and that of the greater is 32 inches. Find the distance of the smaller chord from the centre.

Ex. 48. If D is the middle point of side BC of triangle ABC , right-angled at C , prove $\overline{AB}^2 - \overline{AD}^2 = 3 \overline{CD}^2$.

PROP. XXVI. THEOREM

258. *If any two chords be drawn through a fixed point within a circle, the product of the segments of one chord is equal to the product of the segments of the other.*

Draw a $\odot$; let P be any point within it. Through P draw two chords AB and $A'B'$. We then have:

Given AB and $A'B'$ any two chords passing through fixed point P within $\odot AA'B$.

To Prove $AP \times BP = A'P \times B'P$.

Proof. 1. Draw lines AA' and BB' .

2. $\angle A$ and B' are measured by one-half the same arc, as also are $\angle A'$ and B . (?)

3. $\triangle AA'P$ and $BB'P$ are similar. (§ 236)

4. Find sides of $\triangle BB'P$ homologous to sides AP and $A'P$ of $\triangle AA'P$. (§ 239)

5. Then, $\frac{AP}{A'P} = \frac{B'P}{BP}$; and $AP \times BP = A'P \times B'P$. (?)

Note. If, in the figure of Prop. XXVI, chord AB be supposed to revolve about point P as a pivot, the variable segments of the chord will have a constant product; and if one segment increases, the other decreases in the same ratio.

If, for example, AP were doubled, BP would be halved.

If two variable magnitudes are so related that, if one increases, the other decreases in the same ratio, they are said to be *reciprocally proportional*.

Thus, the segments of a chord by a fixed point are reciprocally proportional; and the theorem may be written:

If any two chords be drawn through a fixed point within a circle, their segments are reciprocally proportional.

Ex. 49. In a triangle ABC , CD is a median and $4\overline{CD}^2 = \overline{AC}^2 + \overline{CB}^2$. Prove that $\overline{AB}^2$ equals $\overline{AC}^2$ plus four times the square of the projection of CD upon CB .

Ex. 50. The sides of a triangle are 21, 20, and 18, respectively. Find the length of the median drawn to the side 18.

Ex. 51. The sides of a triangle are 21, 20, and 18. A median is drawn to the side 18. Find the projection of the median upon this side.

Ex. 52. Using values found in Ex. 50, find the altitude of the above triangle drawn to side 18.

Ex. 53. The median drawn to the base of a triangle is $1\frac{3}{4}$. The other sides of the triangle are 12 and 5. Find the base and altitude.

Ex. 54. Given a circle and a chord CD . Find a point P in the circumference such that $\frac{PC}{PD} = \frac{5}{3}$.

Ex. 55. If, in triangle ABC , the altitudes AD , BE , and CF intersect at O , prove $\frac{OA}{OB} = \frac{OE}{OD}$ and $\frac{OB}{OC} = \frac{OF}{OE}$.

Ex. 56. Two sides of a parallelogram are a and b , respectively. Prove that the bisectors of the angles of the parallelogram form a rectangle, whose diagonal is $b - a$.

Ex. 57. Two chords intersect in a circle. The segments of the first are 8 and 9. One segment of the second is $4\frac{1}{2}$; find the other. Also, if the segments of the first are a and b , and a segment of the second is c .

PROP. XXVII. THEOREM

259. *If through a fixed point without a circle a secant and a tangent be drawn, the product of the whole secant and its external segment is equal to the square of the tangent.*

Draw a $\odot$. Through P , any point without the $\odot$, draw line PC tangent to the $\odot$ at C ; through P draw a secant intersecting the circumference at B , and terminating in the circumference at A . We then have:

Given AP a secant, and CP a tangent, passing through fixed point P without $\odot ABC$.

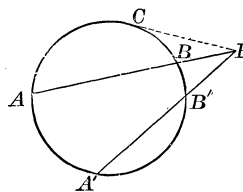
To Prove $AP \times BP = \overline{CP}^2$.

(Draw lines AC , BC . $\angle A = \angle BCP$, for each is measured by $\frac{1}{2}$ arc BC (?); then $\triangle ACP$ and BCP are similar, and their homologous sides are proportional.)

260. It follows from § 259 that *if through a fixed point without a circle a secant and a tangent be drawn, the tangent is a mean proportional between the whole secant and its external segment.*

261. *If any two secants be drawn through a fixed point without a circle, the product of one and its external segment is equal to the product of the other and its external segment.*

For if P be any point without $\odot ABC$, AP and $A'P$ secants intersecting the circumference at A and B , and A' and B' , respectively, and CP tangent to the $\odot$ at C , then



$$AP \times BP = A'P \times B'P;$$

for both $AP \times BP$ and $A'P \times B'P$ equal $\overline{CP}^2$. (§ 259)

Note. If any two secants be drawn through a fixed point without a circle, the entire secants and their external segments are reciprocally proportional. (Compare Note, § 258.)

Ex. 58. Two chords, AB and CD , intersect at K . The centre of the circle is O . $CK = 8$, $KD = 5$, $OK = 3$. Find the radius of the circle.

Ex. 59. A tangent to a circle at a fixed point is 12 feet, and the diameter of the circle is 12 feet. Locate the point from which the tangent is drawn.

Ex. 60. One side of an equilateral triangle is 16; find the altitude, correct to three decimal places.

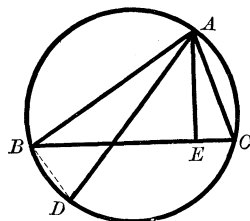
Ex. 61. One side of a square is 1; find the diagonal of the square, correct to three decimal places.

Ex. 62. A tangent 12 inches long and a secant 16 inches long are drawn from an external point to a circle. If the secant passes through the centre, find the radius of the circle.

Ex. 63. If a secant be divided by the circumference in the ratio of 1 to 2, what is the length of the tangent drawn to its extremity?

PROP. XXVIII. THEOREM

262. *In any triangle, the product of any two sides is equal to the diameter of the circumscribed circle, multiplied by the perpendicular drawn to the third side from the vertex of the opposite angle.*



Draw $\triangle ABC$. Circumscribe a $\odot$ about it (§ 206); draw diameter AD , and line $AE \perp BC$. We then have:

Given AD a diameter of the circumscribed $\odot ACD$ of $\triangle ABC$, and line $AE \perp BC$.

To Prove $AB \times AC = AD \times AE$.

(In rt. $\triangle ABD$ and ACE , $\angle D = \angle C$; then, the $\triangle$ are similar, and their homologous sides are proportional.)

263. It follows from § 262 that *in any triangle, the diameter of the circumscribed circle is equal to the product of any two sides divided by the perpendicular drawn to the third side from the vertex of the opposite angle.*

PROP. XXIX. THEOREM

264. *In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.*

Draw $\triangle ABC$, and line AD bisecting $\angle A$, and meeting BC at D . We then have :

Given, in $\triangle ABC$, line AD bisecting $\angle A$, meeting side BC at D .

To Prove $AB \times AC = BD \times DC + AD^2$.

Proof. 1. Circumscribe a $\odot$ about $\triangle ABC$; extend AD to meet the circumference at E , and draw line CE .

2. In $\triangle ABD$ and ACE , $\angle BAD = \angle CAE$, by hyp.

3. Also $\angle B = \angle E$. (?)

4. Then, $\triangle ABD$ and ACE are similar. (?)

5. Find sides of $\triangle ACE$ homologous to sides AB and AD of $\triangle ABD$. (§ 239)

6. Then, $\frac{AB}{AD} = \frac{AE}{AC}$, or $AB \times AC = AD \times AE$. (?)

7. Then, $AB \times AC = AD \times (DE + AD) = AD \times DE + AD^2$.

8. Express $AD \times DE$ in terms of BD and DC . (§ 258)

265. The following cases in which two triangles are *similar* will be found useful in solving original exercises.

Two triangles are similar :

When they have two angles of one equal to two angles of the other (§ 236).

When their homologous sides are proportional (§ 240).

When they have an angle of one equal to an angle of the other, and the sides including these angles proportional (§ 242).

When their sides are parallel each to each, or perpendicular each to each (§ 243).

Two right triangles are similar :

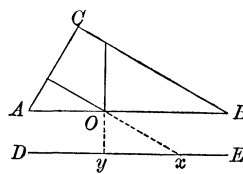
When an acute angle of one is equal to an acute angle of the other (§ 237).

Ex. 64. CD is a common chord of two equal circles. At C tangents are drawn to each circle and meet the circumferences in E and F . Prove $\triangle EDC$ equal to $\triangle DFC$.

Ex. 65. In a circle whose centre is O , CD is a chord perpendicular to a diameter at any point F not the centre. At C a tangent is drawn, and DE perpendicular to the tangent. Prove $\frac{CD}{OC} = \frac{DE}{CF}$.

Ex. 66. AB is a diameter of a circle with centre at O . CD is a chord perpendicular to AB at E . DF is a line through O , and CF is perpendicular to DF . OC is drawn. Prove $\triangle COE$ similar to $\triangle DFC$.

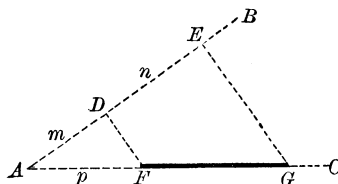
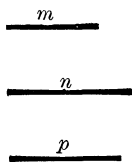
Ex. 67. ABC is a right triangle, O any point in the hypotenuse AB . Through O two perpendiculars are drawn, one to the leg AC , the other to AB . These perpendiculars are extended to meet any line DE parallel to AB , at x and y , respectively. Prove $\frac{xy}{BC} = \frac{Ox}{AB}$.



CONSTRUCTIONS

PROP. XXX. PROBLEM

266. To construct a fourth proportional (§ 215) to three given straight lines.



Given lines m , n , and p .

Required to construct a fourth proportional to m , n , and p .

Construction. 1. Draw lines AB and AC , making any $\angle$.
2. On AB take $AD = m$, and $DE = n$; on AC take $AF = p$.
3. Draw line DF , and line $EG \parallel DF$, meeting AC at G .

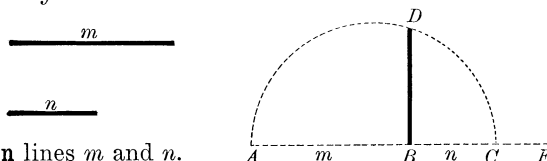
4. Then, $\frac{m}{n} = \frac{p}{FG}$. (?)

267. If $AF = n$, the above proportion becomes $\frac{m}{n} = \frac{n}{FG}$.

In this case, FG is a *third proportional* (§ 214) to m and n .

PROP. XXXI. PROBLEM

268. To construct a mean proportional (§ 214) between two given straight lines.



Given lines m and n .

Required to construct a mean proportional between m and n .

Construction. 1. On line AE , take $AB = m$, and $BC = n$.

2. With AC as a diameter, describe a semi-circumference, and draw line $BD \perp AC$, meeting the arc at D .

3. Then, $\frac{m}{BD} = \frac{BD}{n}$. (§ 250, 1)

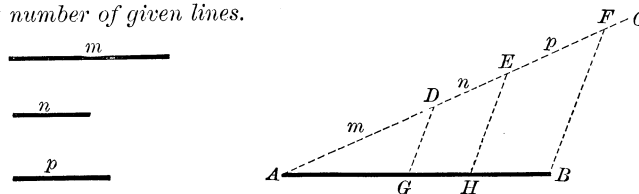
269. By aid of § 268, a line may be constructed equal to $\sqrt{a}$, where a is any number whatever.

Thus, to construct a line equal to $\sqrt{3}$, we take AB equal to 3 units, and BC equal to 1 unit.

Then, $BD = \sqrt{AB \times BC}$ (§ 217) $= \sqrt{3 \times 1} = \sqrt{3}$.

PROP. XXXII. PROBLEM

270. To divide a given straight line into parts proportional to any number of given lines.



Given line AB , and lines m , n , and p .

Required to divide AB into parts proportional to m , n , and p .

Construction. 1. On line AC take $AD = m$, $DE = n$, and $EF = p$.

2. Draw line BF ; and lines DG and $EH \parallel BF$, meeting AB at G and H , respectively.

3. Then,
$$\frac{AG}{m} = \frac{GH}{n} = \frac{HB}{p}. \quad (1)$$

Proof. 4. In $\triangle AEH$, $\frac{AH}{AE} = \frac{AG}{m} = \frac{GH}{n}. \quad (?)$

5. In $\triangle ABF$, $\frac{AH}{AE} = \frac{HB}{p}$; whence, equation (1). $(?)$

Ex. 68. Construct a fourth proportional to three lines whose lengths are 8, 9, and 12, respectively.

Ex. 69. Construct a fourth proportional to 12, 8, and 6.

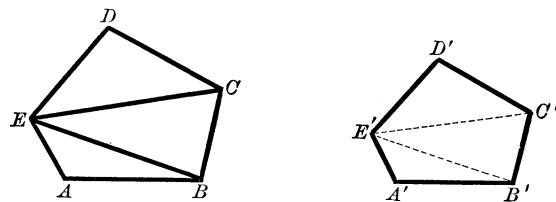
Ex. 70. Construct a third proportional to lines whose lengths are 8 and 6, respectively.

Ex. 71. Construct a mean proportional to lines whose lengths are 9 and 4.

Ex. 72. Construct a line whose length is $\sqrt{5}$.

PROP. XXXIII. PROBLEM

271. Upon a given side, homologous to a given side of a given polygon, to construct a polygon similar to the given polygon.



Given polygon $ABCDE$, and line $A'B'$.

Required to construct upon side $A'B'$, homologous to AB , a polygon similar to $ABCDE$.

Construction. 1. Divide polygon $ABCDE$ into $\triangle$ by drawing diagonals EB and EC .

2. Construct $\triangle A'B'E'$, ABE similar, with $\angle A' = \angle A$, and $\angle A'B'E' = \angle ABE$. $(?)$

3. In like manner, construct $\triangle B'C'E'$ similar to $\triangle BCE$, and $\triangle C'D'E'$ similar to $\triangle CDE$.

272. Def. A straight line is said to be divided by a given point in *extreme and mean ratio* when one of the segments (§ 231) is a mean proportional between the whole line and the other segment.



Thus, line AB is divided *internally* in extreme and mean ratio at C if

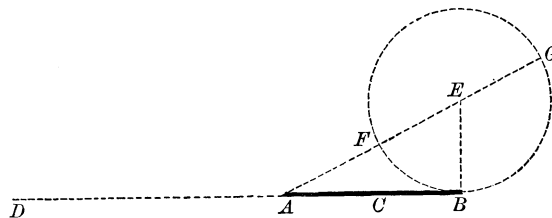
$$\frac{AB}{AC} = \frac{AC}{BC};$$

and *externally* in extreme and mean ratio at D if

$$\frac{AB}{AD} = \frac{AD}{BD}.$$

PROP. XXXIV. PROBLEM

273. To divide a given straight line in extreme and mean ratio (§ 272).



Given line AB .

Required to divide it in extreme and mean ratio.

Construction. 1. With radius $BE = \frac{1}{2} AB$, draw $\odot BFG$ tangent to AB at B ; and line AE cutting circumference at F and G .

2. On AB take $AC = AF$; on BA extended take $AD = AG$; then, AB is divided at C internally, and at D externally, in extreme and mean ratio.

Proof. 3. By § 261, $\frac{AG}{AB} = \frac{AB}{AF}$; or $\frac{AG}{AB} = \frac{AB}{AC}$. (1)

4. Then,
$$\frac{AG - AB}{AB} = \frac{AB - AC}{AC} (?) \quad (2)$$

5. By cons.,
$$AB = 2 BE = FG.$$

6. Then,
$$AG - AB = AG - FG = AF = AC.$$

7. Substitute in (2),
$$\frac{AC}{AB} = \frac{BC}{AC}; \text{ or } \frac{AB}{AC} = \frac{AC}{BC} (\S 220). \quad (3)$$

8. Again from (1),
$$\frac{AG + AB}{AG} = \frac{AB + AC}{AB} (?) \quad (4)$$

9. But
$$AG + AB = AD + AB = BD;$$

and
$$AB + AC = FG + AF = AD. \quad (?)$$

10. Substitute in (4),
$$\frac{BD}{AD} = \frac{AD}{AB}; \text{ or } \frac{AD}{BD} = \frac{AB}{AD}.$$

274. Putting $AB = m, AC = x, BC = m - x$ in (3),

$$\frac{m}{x} = \frac{x}{m - x}; \text{ or } x^2 = m(m - x). \quad (\S 216)$$

Then,
$$x^2 = m^2 - mx; \text{ or } x^2 + mx = m^2.$$

Multiplying by 4, and adding m^2 to both members,

$$4x^2 + 4mx + m^2 = 4m^2 + m^2 = 5m^2.$$

Extracting the square root of both members,

$$2x + m = \pm m\sqrt{5}.$$

Since x cannot be negative, we take the positive sign before the radical sign; then,

$$2x = m\sqrt{5} - m; \text{ and } x \text{ (or } AC) = \frac{m(\sqrt{5} - 1)}{2}.$$

BOOK IV

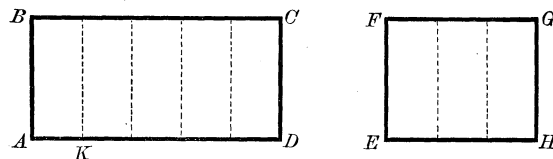
AREAS OF POLYGONS

PROP. I. THEOREM

275. *Two rectangles having equal altitudes are to each other as their bases.*

The words “rectangle,” “parallelogram,” “triangle,” etc., in the propositions of Book IV, mean the *amount of surface* in the rectangle, parallelogram, triangle, etc.

Case I. *When the bases are commensurable.*



Draw rectangles $ABCD$ and $EFGH$ having equal altitudes and the bases AD and EH commensurable. We then have :

Given rectangles $ABCD$ and $EFGH$, with equal altitudes AB and EF , and commensurable bases AD and EH .

To Prove
$$\frac{ABCD}{EFGH} = \frac{AD}{EH}. \quad (1)$$

Proof. 1. Let AK be contained 5 times in AD , and 3 times in EH .

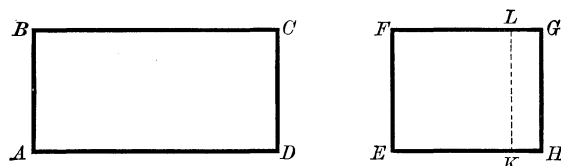
2. Find ratio of AD to EH .

3. Through points of division of AD and EH draw $\perp$ s to these lines.

4. Find ratio of $ABCD$ to $EFGH$. (§ 111)

5. Prove equation (1). (?)

Case II. *When the bases are incommensurable.*



Draw rectangles in accordance with the statement. We then have :

Given rectangles $ABCD$ and $EFGH$, with equal altitudes AB and EF , and incommensurable bases AD and EH .

To Prove
$$\frac{ABCD}{EFGH} = \frac{AD}{EH}. \quad (2)$$

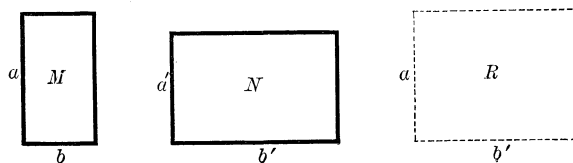
- Proof.**
1. Divide AD into any number of equal parts.
 2. Let one of these parts be contained exactly in EK , with a remainder $KH <$ one of the parts.
 3. Draw line $KL \perp EH$, meeting FG at L .
 4. By Case I,
$$\frac{ABCD}{EFLK} = \frac{AD}{EK}.$$
 5. Let number of subdivisions of AD be indefinitely increased.
 6. Find limits $\frac{ABCD}{EFLK}$ and $\frac{AD}{EK}$.
 7. Prove equation (2) by Theorem of Limits.

276. Since either side of a rectangle may be taken as the base, it follows that

Two rectangles having equal bases are to each other as their altitudes.

PROP. II. THEOREM

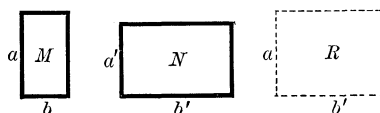
277. *Any two rectangles are to each other as the products of their bases by their altitudes.*



Draw any two rectangles M and N . We then have :

Given M and N rectangles, with altitudes a and a' , and bases b and b' respectively.

To Prove $\frac{M}{N} = \frac{a \times b}{a' \times b'}$. (1)



Proof. 1. Let R be a rect. with altitude a and base b' .

2. M and R have = altitudes; and bases b and b' , respectively; find ratio M to R by § 275.

3. R and N have = bases; and altitudes a and a' , respectively; find ratio R to N by § 276.

4. Multiplying results of steps (2) and (3), gives (1).

DEFINITIONS

278. The *area* of a surface is its ratio to another surface, called the *unit of surface*, adopted arbitrarily as the unit of measure (§ 180).

The usual unit of surface is a square whose side is some *linear unit*; for example, a *square inch* or a *square foot*.

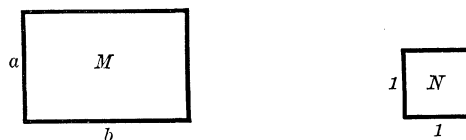
279. We call two surfaces *equivalent* ($\asymp$) when their areas are equal.

The *dimensions* of a rectangle are its base and altitude.

PROP. III. THEOREM

280. *The area of a rectangle is equal to the product of its base and altitude.*

In all propositions relating to areas, the unit of surface (§ 278) is understood to be a square whose side is the linear unit.



Draw any rectangle M , and a square N whose side is 1. We then have :

Given a the altitude and b the base, of rect. M ; and N the unit of surface, a square whose side is the linear unit.

To Prove that, if N is the unit of surface,

$$\text{area } M = a \times b. \quad (1)$$

Proof. 1. We have $\frac{M}{N} = \frac{a \times b}{1 \times 1}$. (§ 277)

2. N being unit of surface, $\frac{M}{N}$ is *area* of M ; whence equation (1).

Note. The statement of Prop. III is an abbreviation of the following:

If the unit of surface is a square whose side is the linear unit, the *number* which expresses the area of a rectangle is equal to the product of the *numbers* which express the lengths of its sides.

An interpretation of this form is always understood in every proposition relating to areas.

281. By § 280, *the area of a square equals the square of its side.*

Ex. 1. Find the ratio of the area of a square to the product of its diagonals.

Ex. 2. The area of a rectangle whose base is 24 is 432. Find the diagonal of the rectangle.

Ex. 3. How many shingles will it take to cover a roof 36 feet by 18 feet (one side) with shingles that average 4 inches wide, 16 inches long, and are laid 6 inches to the weather?

PROP. IV. THEOREM

282. *The area of a parallelogram is equal to the product of its base and altitude.*

Draw $\square ABCD$, AD (b) being the base; draw DF (a) the altitude meeting BC at F . We then have:

Given b the base, and a the altitude, of $\square ABCD$.

To Prove $\text{area } ABCD = a \times b$. (1)

Proof. 1. Draw line $AE \parallel DF$, meeting CB prolonged at E .

2. Then, rt. $\triangle ABE$ and DCF are equal. (§§ 61, 104)

3. If from figure $ADCE$ we take $\triangle ABE$, there remains $\square AC$; if we take $\triangle DCF$, there remains rect. AF .

4. Then, $\text{area } ABCD = \text{area } AEFD$; whence equation (1).

283. It follows from § 282 that:

1. *Two parallelograms having equal bases and equal altitudes are equivalent (§ 279).*
2. *Two parallelograms having equal altitudes are to each other as their bases.*
3. *Two parallelograms having equal bases are to each other as their altitudes.*
4. *Any two parallelograms are to each other as the products of their bases by their altitudes.*

Ex. 4. The area of a parallelogram is 288, the base is twice the altitude. Find the dimensions. Construct the parallelogram. Can more than one such parallelogram be drawn? How many and what parts are necessary for a definite figure? Why?

Ex. 5. Find the ratio of the area of a rhombus to the product of its diagonals.

PROP. V. THEOREM

284. *The area of a triangle is equal to one-half the product of its base and altitude.*

Draw $\triangle ABC$ having base BC (b); draw $AE \perp BC$, meeting BC , or BC extended, at E . We then have:

Given b the base, and a the altitude of $\triangle ABC$.

To Prove $\text{area } ABC = \frac{1}{2} a \times b$.

(Draw line $AD \parallel BC$, and line $CD \parallel AB$. By § 105 AC divides $\square ABCD$ into two equal $\triangle$.)

285. It follows from § 284 that:

1. *Two triangles having equal bases and equal altitudes are equivalent.*
2. *Two triangles having equal altitudes are to each other as their bases.*
3. *Two triangles having equal bases are to each other as their altitudes.*
4. *Any two triangles are to each other as the products of their bases by their altitudes.*

5. *The area of any triangle is one-half that of a parallelogram having the same base and altitude.*

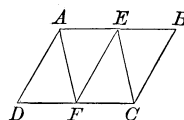
Ex. 6. Draw a line dividing a given right triangle into two equivalent isosceles triangles.

Ex. 7. Prove that each of its medians divides a triangle into two equivalent parts.

Ex. 8. The base of a triangle is 37 feet, the altitude is 32 feet. How many square yards in its area?

Ex. 9. The area of a triangle is 216, the altitude is 12; find the base.

Ex. 10. If E and F are the middle points of sides AB and CD , respectively, of parallelogram $ABCD$, the lines AF , EF , and CE divide the parallelogram into four equal triangles.



Ex. 11. If the middle point of any side of a parallelogram be joined to the opposite vertices, the triangle included by these lines and the opposite side is equivalent to one-half the parallelogram.

Ex. 12. If the middle point of a diagonal of any quadrilateral be joined to the opposite vertices, the quadrilateral is divided into two pairs of equivalent triangles, and into two equivalent parts.

PROP. VI. THEOREM

286. *The area of a trapezoid is equal to one-half the sum of its bases multiplied by its altitude.*

Draw trapezoid $ABCD$, AB (b) and DC (b') being $\parallel$ sides. From D draw altitude DE (a) meeting AB at E . We then have:

Given AB (b) and DC (b') the bases, and DE (a) the altitude, of trapezoid $ABCD$.

To Prove $\text{area } ABCD = a \times \frac{1}{2}(b + b')$.

(Draw diagonal BD . The trapezoid is composed of two $\triangle$ whose altitude is a , and bases b and b' , respectively.)

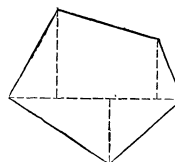
287. Since the line joining the middle points of the non-parallel sides of a trapezoid is equal to one-half the sum of the bases (§ 132), it follows that

The area of a trapezoid is equal to the product of its altitude by the line joining the middle points of its non-parallel sides.

Note. The area of any polygon may be obtained by finding the sum of the areas of the triangles into which the polygon may be divided by drawing diagonals from any one of its vertices.

But in practice it is better to draw the longest diagonal, and draw perpendiculars to it from the remaining vertices of the polygon.

The polygon will then be divided into right triangles and trapezoids; and by measuring the lengths of the perpendiculars, and of the portions of the diagonal which they intercept, the areas of the figures may be found by §§ 284 and 286.



PROP. VII. THEOREM

288. *Two similar triangles are to each other as the squares of their homologous sides.*

Draw $\triangle ABC$; and construct $\triangle A'B'C'$ similar to $\triangle ABC$, AB (c) and $A'B'$ (c') being homologous sides. We then have :

Given c and c' homologous sides of $\triangle ABC$ and $A'B'C'$, respectively.

To Prove
$$\frac{ABC}{A'B'C'} = \frac{c^2}{c'^2}. \quad (1)$$

Proof. 1. Draw altitudes $CD(h)$ and $C'D'(h') \perp AB$ and $A'B'$, respectively.

2. Find ratio ABC to $A'B'C'$ in terms of h and c , and h' and c' . (§ 285, 4)

3. Express ratio h to h' in terms of c and c' . (§ 245)

4. Substituting in result of step 2, gives (1).

289. Note. Two similar triangles are to each other as the squares of any two homologous lines.

Ex. 13. A field has two sides parallel, the lengths of these sides being 160 yards and 302 yards, respectively. The distance between the parallel sides is 48 rods. How many acres has the field? Draw a diagram.

Ex. 14. A field has two parallel sides whose lengths are 322 and 168 yards, respectively. The area of the field is 12 acres. Find the distance between the parallel sides and make a diagram of the field.

Ex. 15. A square field contains 3 acres 12 square rods. Find its perimeter in rods, correct to three decimal places.

Ex. 16. A square field contains one-half a square mile. Find its diagonal in rods, correct to three decimal places.

Ex. 17. The base of a rectangle is 462 feet. One diagonal is 504 feet; find the area of the rectangle.

Ex. 18. One-quarter of a government tree claim, containing 160 acres, is to be planted with trees 4 feet apart each way. How many trees will be required? Will the same number of trees be required whether the part selected for planting be rectangular or square?

Ex. 19. The area of a trapezoid is 2774, the altitude is 38, the upper base is 29; find the lower base.

Ex. 20. One of the equal sides of an isosceles trapezoid is 10 feet, the lower base is 12 feet longer than the upper base; find the altitude.

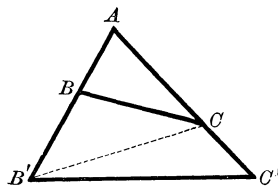
Ex. 21. One side of a rhombus is 20, the longer diagonal is 32; find the other diagonal, the area, and the altitude.

Ex. 22. If from any point within an equilateral triangle perpendiculars to the sides be drawn, the area of the triangle is equal to one-half the sum of the perpendiculars multiplied by one side of the triangle.

Ex. 23. Find the area of a trapezoid whose parallel sides are a and b , and whose altitude is 2, divided by the sum of the parallel sides. If a and b vary, will the area vary? Will the form of the trapezoid change? Draw the trapezoid.

PROP. VIII. THEOREM

290. *Two triangles having an angle of one equal to an angle of the other, are to each other as the products of the sides including the equal angles.*

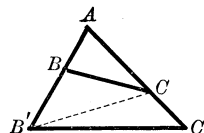


Draw $\triangle AB'C'$ and line BC meeting AB' at B , and AC' at C . We then have :

Given $\angle A$ common to $\triangle ABC$ and $AB'C'$.

To Prove $\frac{ABC}{AB'C'} = \frac{AB \times AC}{AB' \times AC'}. \quad (1)$

Proof. 1. Draw line $B'C$; $\triangle ABC$ and $AB'C$ have same vertex C , and bases AB and AB' in same str. line.



2. Express ratio ABC to $AB'C$ in terms of AB and AB' .

(§ 285, 2)

3. $\triangle AB'C$ and $AB'C'$ have same vertex B' , and bases AC and AC' in same str. line.

4. Express ratio of $AB'C$ to $AB'C'$ in terms of AC and AC' .

5. Multiplying results of steps (2) and (4) gives equation (1).

Ex. 24. If the equal angles of an equilateral triangle be bisected, the bisectors and the base form a triangle whose area is one-third that of the given triangle.

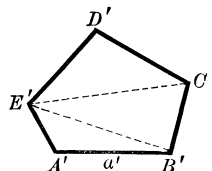
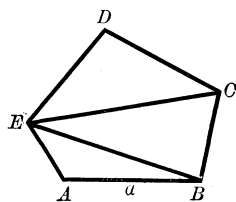
Ex. 25. Any two altitudes of a triangle are inversely as the sides to which they are drawn.

Ex. 26. The altitude of a triangle is 8, its area is 48, and the perpendicular to the base divides the base in the ratio of 2 to 1. Find the sides of the triangle. Construct the triangle.

Ex. 27. Two triangles have equal areas and equal vertical angles. Prove that the products of the sides including the equal angles are equal.

PROP. IX. THEOREM

291. *Two similar polygons are to each other as the squares of their homologous sides.*



Draw polygon $ABCDE$; construct polygon $A'B'C'D'E'$ similar to $ABCDE$, AB (a) and $A'B'$ (a') being homologous sides. We then have:

Given a and a' homologous sides of similar polygons AC and $A'C'$, whose areas are K and K' , respectively.

To Prove
$$\frac{K}{K'} = \frac{a^2}{a'^2}. \quad (1)$$

Proof. 1. Draw diagonals from E and E' .

2. $\triangle ABE$ is similar to $\triangle A'B'E'$. (§ 247)

3. Express ratio ABE to $A'B'E'$ in terms of a and a' . (§ 288)

4. Also,
$$\frac{BCE}{B'C'E'} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{a^2}{a'^2}; \quad \frac{CDE}{C'D'E'} = \frac{\overline{CD}^2}{\overline{C'D'}^2} = \frac{a^2}{a'^2}. \quad (\S 234, 2)$$

5. Then,
$$\frac{ABE}{A'B'E'} = \frac{BCE}{B'C'E'} = \frac{CDE}{C'D'E'}. \quad (?)$$

6. Apply § 225 to above result.

7. Substituting values of members in above result gives (1).

Ex. 28. The area of a polygon having a side 6 feet is 112 square feet; find the area of a similar polygon whose homologous side is 9 feet.

Ex. 29. The ratio of the homologous sides of two similar polygons is 2 to 3. What is the ratio of their areas? Find a side of the first. If the area of the second is A , what is the area of the first?

Ex. 30. The area of a polygon is $\frac{15}{16}$ the area of a similar polygon; find the ratio of their homologous sides.

PROP. X. PROBLEM

292. *To find the area of a triangle in terms of its sides.*

Draw $\triangle ABC$, a , b , c , being the sides opposite $\angle A$, B , and C , respectively, and $\angle C$ acute. We then have:

Given sides a , b , and c of $\triangle ABC$.

Required to find area ABC in terms of a , b , and c .

Solution. 1. Draw altitude AD .

2. Then,
$$c^2 = a^2 + b^2 - 2a \times CD. \quad (\S 255)$$

3. Transposing,
$$2a \times CD = a^2 + b^2 - c^2,$$

or
$$CD = \frac{a^2 + b^2 - c^2}{2a}.$$

$$\begin{aligned}
4. \text{ Then, } \overline{AD}^2 &= \overline{AC}^2 - \overline{CD}^2 & (\S 253) \\
&= (AC + CD)(AC - CD) \\
&= \left(b + \frac{a^2 + b^2 - c^2}{2a}\right) \left(b - \frac{a^2 + b^2 - c^2}{2a}\right) \\
&= \frac{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)}{4a^2} \\
&= \frac{[(a+b)^2 - c^2][c^2 - (a-b)^2]}{4a^2} \\
&= \frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2}. \quad (1)
\end{aligned}$$

5. Now let $a + b + c = 2s$.

$$\begin{aligned}
6. \text{ Then, } \overline{AD}^2 &= \frac{2s(2s-2c)(2s-2b)(2s-2a)}{4a^2} \\
&= \frac{16s(s-a)(s-b)(s-c)}{4a^2}.
\end{aligned}$$

$$7. \text{ Then, } AD = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}.$$

$$\begin{aligned}
8. \text{ Then, } \text{area } ABC &= \frac{1}{2}a \times AD & (?) \\
&= \sqrt{s(s-a)(s-b)(s-c)}.
\end{aligned}$$

As an example of § 292, let it be required to find the area of a triangle whose sides are 13, 14, and 15.

Let $a = 13$, $b = 14$, and $c = 15$; then

$$s = \frac{1}{2}(13 + 14 + 15) = 21.$$

Whence, $s - a = 8$, $s - b = 7$, and $s - c = 6$.

Then, the area of the triangle is

$$\begin{aligned}
\sqrt{21 \times 8 \times 7 \times 6} &= \sqrt{3 \times 7 \times 2^3 \times 7 \times 2 \times 3} \\
&= \sqrt{2^4 \times 3^2 \times 7^2} = 2^2 \times 3 \times 7 = 84.
\end{aligned}$$

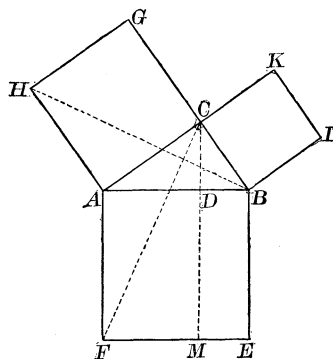
Ex. 31. If a line be drawn from the middle point of each side of a quadrilateral to the middle point of the next side in order, the figure formed is a parallelogram whose area is one-half that of the quadrilateral.

Ex. 32. Prove that the lines connecting the middle points of the sides of a triangle divide it into four equal triangles.

293. Note. Since the area of a square is equal to the square of its side (§ 281), we may state Prop. XXII, Book III, as follows:

In any right triangle, the square described upon the hypotenuse is equivalent to the sum of the squares described upon the legs.

The theorem in the above form may be proved as follows:



Draw $\triangle ABC$, right-angled at C ; on AB , AC , and BC describe squares $ABEF$, $ACGH$, and $BCKL$, respectively. We then have:

Given squares AE , AG , and BK described upon the hypotenuse and legs of $\triangle ABC$.

To Prove $ABEF \approx ACGH + BCKL$. (1)

Proof. 1. Draw line $CD \perp AB$, and prolong it to meet EF at M ; also, lines BH and CF .

2. We have $\triangle ABH = \triangle ACF$; since $AB = AF$, $AH = AC$, and $\angle BAH = \angle CAF$, each being a rt. $\angle + \angle BAC$. (?)

3. Then $ABH \approx \frac{1}{2} ACGH$, having same base and altitude. (§ 285, 5)

4. Again, $ACF \approx \frac{1}{2} ADMF$.

5. Then $\frac{1}{2} ACGH \approx \frac{1}{2} ADMF$, or $ACGH \approx ADMF$.

6. Similarly, drawing lines from A to L and from C to E , $BCKL \approx BDME$.

7. Adding results of (5) and (6), gives equation (1).

Note. The theorem of § 293 is supposed to have been first given by Pythagoras, and is called after him the *Pythagorean Theorem*.

Several other propositions of Book III may be put in the form of statements in regard to areas; for example, Props. XXIII and XXIV.

Ex. 33. In a triangle whose sides are a , b , and c , the angle opposite side c is 90° . Find the area in terms of the sides.

Ex. 34. The sides of a triangle are $a + b$, $b + c$, $c + a$; find the altitude to side $a + b$.

Ex. 35. The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the four sides of the parallelogram. [Project one line upon another.]

Ex. 36. Given a trapezoid with bases b_1 and b_2 , altitude h and area S , find the altitude a of an equivalent equilateral triangle whose area is S_2 . [$S_1 = S_2$.]

Ex. 37. If lines be drawn through the extremities of the diagonals of a rhombus parallel to the diagonals, they form with the diagonals four rectangles, each equivalent to one-half the rhombus.

Ex. 38. The sides of a rectangle are 24 and 36. The altitude of a triangle of the same area is 33; find the base to which this altitude is drawn. Can more than one triangle be drawn which satisfies the conditions of the problem?

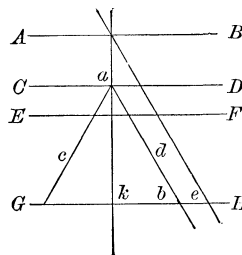
Ex. 39. AB , CD , EF , GH are parallel lines. EF is equidistant from AB and GH . State and prove all theorems suggested by the figure.

Ex. 40. Given two chords AB and AC in a circle of 5-inch radius, such that central angle $AOB = 60^\circ$, and OB is perpendicular to AC , to find the distance of each chord from the centre O .

Ex. 41. If D is the intersection of the perpendiculars from the vertices of triangle ABC to the opposite sides, prove

$$\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2.$$

Ex. 42. Draw figure and give a geometrical proof of the algebraic theorem: The square of the sum of two numbers equals the square of the first, plus twice the product of the first by the second, plus the square of the second.



Ex. 43. Find the ratio of the areas of two triangles which have two sides of one equal to two sides of the other, and the angles formed by the equal sides supplementary.

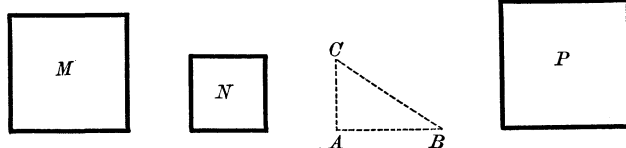
Ex. 44. The square of the altitude of an equilateral triangle is three-fourths the square of one side.

Ex. 45. A person in a train observes that the direction in which the rain appears to fall makes an angle of 60° with the direction of the train's motion, which is at the rate of 45 miles an hour. The rain is really falling vertically. What is its velocity?

CONSTRUCTIONS

PROP. XI. PROBLEM

294. To construct a square equivalent to the sum of two given squares.



Given squares M and N .

Required to construct a square $\approx M + N$.

Construction. Construct a rt. $\triangle$ with legs equal to the sides of M and N , and square P with its side equal to the hypotenuse; then, $P \approx M + N$.

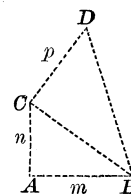
(Prove by § 293.)

295. By an extension of § 293, a square may be constructed equivalent to the sum of any number of given squares.

Thus, suppose we have given *three* squares whose sides are m , n , and p , respectively.

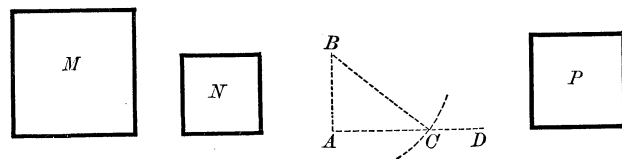
Take line $AB = m$; draw line $AC \perp AB$, and equal to n , and line BC ; draw line $CD \perp BC$, and equal to p , and line BD .

Then, the square having its side equal to BD will be $\approx$ the sum of the given squares.



PROP. XII. PROBLEM

296. *To construct a square equivalent to the difference of two given squares.*



Given squares M and N , $M > N$.

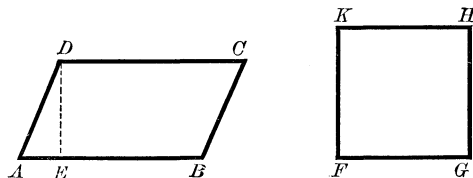
Required to construct a square $\approx M - N$.

Construction. Draw rt. $\angle BAD$, with AB equal to a side of N ; find point C in AD , its distance from B equal to a side of M ; then, a square with side equal to AC will be $\approx M - N$.

(Prove by §§ 253 and 281.)

PROP. XIII. PROBLEM

297. *To construct a square equivalent to a given parallelogram.*



Given $\square ABCD$.

Required to construct a square $\approx ABCD$.

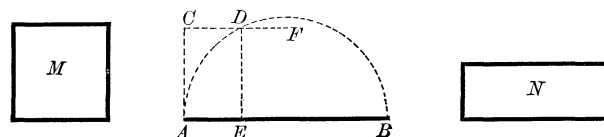
Construction. A square having for its side FG a mean proportional between AB and DE (§ 268), the altitude of the $\square$, will be $\approx ABCD$. (By § 217, $\overline{FG}^2 = AB \times DE$.)

298. *A square may be constructed equivalent to a given triangle by taking for its side a mean proportional between the base and one-half the altitude of the triangle.*

Ex. 46. To find a point within a parallelogram through which if any line be drawn, the parallelogram is divided into two equivalent parts. Are these parts parallelograms? Trapezoids? Triangles?

PROP. XIV. PROBLEM

299. To construct a rectangle equivalent to a given square, having the sum of its base and altitude equal to a given line.



Given square M , and line AB .

Required to construct a rectangle $\approx M$, having the sum of its base and altitude equal to AB .

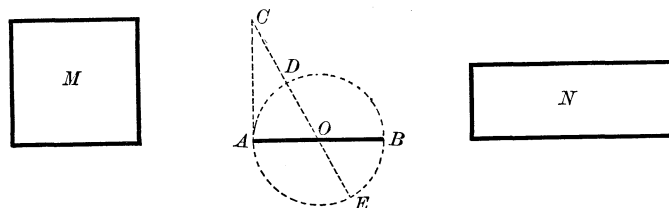
Construction. 1. Construct semi-circumference with diameter AB ; and find point D in its arc by drawing $\parallel$ to AB at a distance from AB equal to a side of M .

2. Draw line $DE \perp AB$; then rectangle with dimensions AE and BE will be $\approx M$.

(By § 250, 1, $\frac{AE}{DE} = \frac{DE}{BE}$; and $\overline{DE}^2 = AE \times BE$.)

PROP. XV. PROBLEM

300. To construct a rectangle equivalent to a given square, having the difference of its base and altitude equal to a given line.



Given square M , and line AB .

Required to construct a rectangle $\approx M$, having the difference of its base and altitude equal to AB .

Construction. 1. Describe $\odot$ with diameter AB .

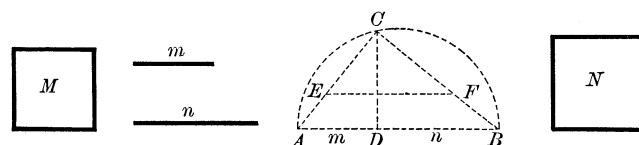
2. Draw tangent AC equal to a side of M .

3. Through C and centre of $\odot$ draw a secant, and construct a rectangle with dimensions equal to the secant and its exterior segment.

(By § 259, $CD \times CE = \overline{CA}^2$; also, $AB = CE - CD$.)

PROP. XVI. PROBLEM

301. To construct a square having a given ratio to a given square.



Given square M , and lines m and n .

Required to construct a square having to M the ratio $\frac{n}{m}$.

Construction. 1. Take $AD = m$, $BD = n$; with AB as diameter describe a semi-circumference; find point C where $\perp$ at D meets arc.

2. On CA take CE equal to a side of M ; at E draw $\parallel$ to AB meeting CB at F ; square N with side equal to CF will be required square.

Proof. 3. We have $\frac{CF}{CE} = \frac{BC}{AC}$; then $\frac{\overline{CF}^2}{\overline{CE}^2} = \frac{\overline{BC}^2}{\overline{AC}^2}$. (1)

4. In rt. $\triangle ABC$, $\frac{\overline{BC}^2}{\overline{AC}^2} = \frac{AB \times BD}{AB \times AD} = \frac{BD}{AD} = \frac{n}{m}$. (§ 250, 2)

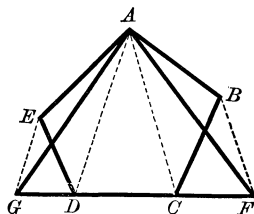
5. Substitute this in (1); and for $\overline{CF}^2$ and $\overline{CE}^2$ areas N and M .

Ex. 47. To inscribe a parallelogram in an equilateral triangle, one angle of the parallelogram being 120° . Can more than one such parallelogram be drawn?

Ex. 48. To inscribe a rhombus in a given triangle, the rhombus and triangle to have one angle in common.

PROP. XVII. PROBLEM

302. *To construct a triangle equivalent to a given polygon.*



Given polygon $ABCDE$.

Required to construct a $\triangle \approx ABCDE$.

Construction. 1. Take three consecutive vertices, A, B, C ; draw diagonal AC , and line $BF \parallel AC$, meeting DC prolonged at F ; also line AF .

2. $\triangle ACF \approx \triangle ABC$; then, polygon $AFDE \approx ABCDE$ (having common part $ACDE$), and has a number of sides less by 1.

3. Draw diagonal AD , and line $EG \parallel AD$, meeting CD prolonged at G , also line AG ; then, $\triangle AED \approx \triangle AGD$, whence $\triangle AFG \approx$ polygon $AFDE$, or to $ABCDE$.

Proof. 4. $\triangle ACF$ and ABC have same base and altitude. (§ 96)

5. Then $\text{area } ACF = \text{area } ABC$. (?)

6. Similarly, $\text{area } AED = \text{area } AGD$.

Note. By aid of §§ 302 and 298, a square may be constructed equivalent to a given polygon.

Ex. 49. The bases of a trapezoid are 8 and 10, respectively, the altitude 6. Construct an equivalent equilateral triangle.

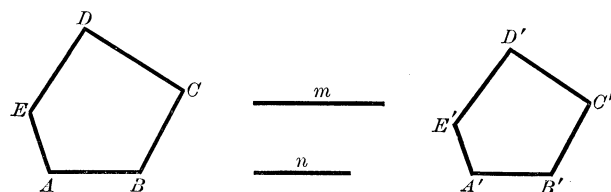
[An equilateral triangle can be constructed when its altitude is known.]

Ex. 50. The bases of a trapezoid are 6 and 8, respectively; the area is 35. Construct the trapezoid and an equivalent equilateral triangle.

Ex. 51. Given $\triangle ABC$. Draw a line from A to BC which shall divide the triangle into two triangles the ratio of whose areas is 1 to 2.

PROP. XVIII. PROBLEM

303. To construct a polygon similar to a given polygon, and having a given ratio to it.



Given polygon AC , and lines m and n .

Required to construct a polygon similar to AC , and having to it the ratio $\frac{n}{m}$.

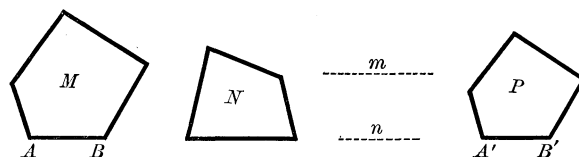
Construction. 1. Let $A'B'$ be side of square having to the square described on AB the required ratio. (§ 301)

2. Polygon $A'C'$, similar to AC , will have required ratio to it.

(By § 291, $\frac{\text{area } A'C'}{\text{area } AC} = \frac{A'B'^2}{AB^2}$, which equals $\frac{n}{m}$.)

PROP. XIX. PROBLEM

304. To construct a polygon similar to one of two given polygons, and equivalent to the other.



Given polygons M and N .

Required to construct a polygon similar to M , and $\approx N$.

Construction. 1. Find m and n sides of squares $\approx M$ and N .
(Note, § 302)

2. Polygon P , similar to M , with $A'B'$ a fourth proportional to m , n , and AB , will be $\approx N$.

Proof. 3. By § 291, $\frac{\text{area } M}{\text{area } P} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{m^2}{n^2}$, by cons.

4. Then $\frac{\text{area } M}{\text{area } P} = \frac{\text{area } M}{\text{area } N}$, and $\text{area } P = \text{area } N$.

Ex. 52. To inscribe in a given triangle a parallelogram whose area is one-half the area of the triangle. Can a parallelogram be inscribed whose area is still greater?

Ex. 53. Through a given point P , either within or without a given angle, to draw a line which shall form with the sides of the angle a triangle of given area.

Ex. 54. To construct a triangle whose angles shall be equal, respectively, to the angles of a given triangle, and whose area shall be four times the area of the given triangle.

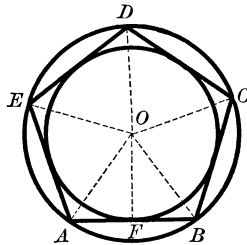
BOOK V

REGULAR POLYGONS.—MEASUREMENT OF THE CIRCLE.—LOCI

305. Def. A *regular polygon* is a polygon which is both equilateral and equiangular.

PROP. I. THEOREM

306. *A circle can be circumscribed about any regular polygon.*



Given regular polygon $ABCDE$.

To Prove that a $\odot$ can be circumscribed about it.

Proof. 1. Draw circumference through A, B, C ; also, radii OA, OB, OC , and line OD .

2. In $\triangle OAB, OCD, OB = OC, AB = CD.$ (?)

3. $\angle OBA = \angle ABC - \angle OBC, \angle OCD = \angle BCD - \angle OCB.$

4. Since $\angle ABC = \angle BCD$ (?), and $\angle OBC = \angle OCB$ (?),
 $\angle OBA = \angle OCD.$

5. Then $\triangle OAB = \triangle OCD$, and $OA = OD.$ (?)

6. Then circumference through A, B, C passes through D ; and similarly through E .

307. Since AB, BC, CD , etc., are equal chords of the circumscribed $\odot$, they are equally distant from O . (§ 164)

Then, a $\odot$ drawn with O as a centre, and a line $OF \perp$ to any side AB as a radius, will be *inscribed* in $ABCDE$.

Hence, *a circle can be inscribed in any regular polygon.*

308. Defs. The *centre* of a regular polygon is the common centre of the circumscribed and inscribed circles.

The *angle at the centre* is the angle between the radii drawn to the extremities of any side; as AOB .

The *radius* is the radius of the circumscribed circle, OA .

The *apothem* is the radius of the inscribed circle, OF .

309. From equal $\triangle OAB, OBC$, etc., we have

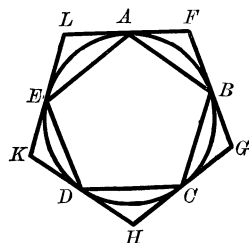
$$\angle AOB = \angle BOC = \angle COD, \text{ etc.} \quad (?)$$

But the sum of these $\angle$ s is four rt. $\angle$ s. (§ 19)

Whence, *the angle at the centre of a regular polygon is equal to four right angles divided by the number of sides.*

PROP. II. THEOREM

310. *If a circumference be divided into equal arcs, their chords form a regular inscribed polygon.*



Given circumference ACD divided into five equal arcs, AB, BC, CD , etc., and chords AB, BC , etc.

To Prove polygon $ABCDE$ regular.

Proof. 1. Sides of polygon are equal by § 159.

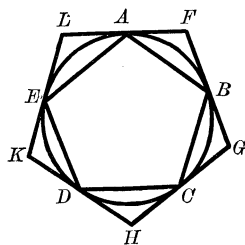
2. Each $\angle$ is measured by one-half the sum of three of the equal arcs. (§ 192)

311. Let lines LF , FG , etc., be tangent to $\odot ACD$ at A , B , etc., respectively, forming polygon $FGHKL$.

In $\triangle ABF$, BCG , CDH , etc., $AB = BC = CD$, etc. (§ 159)

Also, since arc $AB = \text{arc } BC = \text{arc } CD$, etc., we have

$$\angle BAF = \angle ABF = \angle CBG = \angle BCG, \text{ etc.} \quad (\S 196)$$



Whence, ABF , BCG , etc., are equal isosceles $\triangle$. (§§ 49, 90)

Then, $\angle F = \angle G = \angle H$, etc.,

and $BF = BG = CG = CH$, etc. (§ 48)

Then, $FG = GH = HK$, etc.,

and polygon $FGHKL$ is regular.

Then, if a circumference be divided into equal arcs, tangents at the points of division form a regular circumscribed polygon.

312. It follows from §§ 310 and 311 that

1. If from the middle point of each arc subtended by a side of a regular inscribed polygon lines be drawn to its extremities, a regular inscribed polygon of double the number of sides is formed.

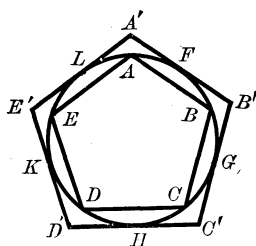
2. If at the middle point of each arc included between two consecutive points of contact of a regular circumscribed polygon tangents be drawn, a regular circumscribed polygon of double the number of sides is formed.

3. An equilateral polygon inscribed in a circle is regular.

For its sides subtend equal arcs. (?)

PROP. III. THEOREM

313. *Tangents to a circle at the middle points of the arcs subtended by the sides of a regular inscribed polygon, form a regular circumscribed polygon.*



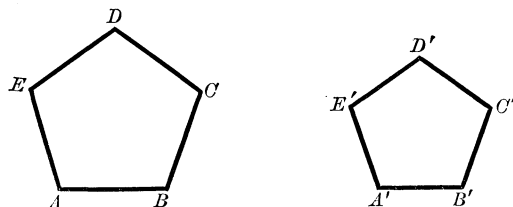
Given $ABCDE$ a regular polygon inscribed in $\odot AC$, and polygon $A'B'C'D'E'$ with sides $A'B'$, $B'C'$, etc., tangent to $\odot AC$ at middle points F , G , etc., of arcs AB , BC , etc., respectively.

To Prove $A'B'C'D'E'$ a regular polygon.

(Arc AF = arc BF = arc BG = arc CG , etc.; use § 311.)

PROP. IV. THEOREM

314. *Regular polygons of the same number of sides are similar.*



Given AC and $A'C'$ regular polygons of five sides.

To Prove AC and $A'C'$ similar. (Use § 233.)

Find the angle, and the angle at the centre,

Ex. 1. Of a regular pentagon.

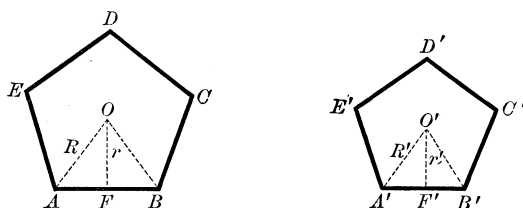
Ex. 2. Of a regular dodecagon.

Ex. 3. Of a regular polygon of 32 sides.

Ex. 4. Of a regular polygon of 25 sides.

PROP. V. THEOREM

315. *The perimeters of two similar regular polygons are to each other as their radii, or as their apothems.*



Given O, O' the centres, OA (R), $O'A'$ (R') the radii, OF (r), $O'F'$ (r') the apothems, and P, P' the perimeters, respectively, of similar regular polygons AC and $A'C'$.

To Prove
$$\frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}. \quad (1)$$

Proof. 1. Draw lines $OB, O'B'$; then $\angle AOB = \angle A'O'B'$. (§ 309)

2. Since $\frac{R}{R'} = \frac{OB}{O'B'}$, $\triangle OAB, \triangle O'A'B'$ are similar. (§ 242)

3. Then,
$$\frac{AB}{A'B'} = \frac{R}{R'} = \frac{r}{r'}. \quad (§ 245)$$

4. Now
$$\frac{AB}{A'B'} = \frac{P}{P'}. \quad (§ 248)$$

5. Substituting in result of (3) gives equation (1).

316. If K denote the area of polygon AC , and K' of $A'C'$,

$$\frac{K}{K'} = \frac{AB^2}{A'B'^2}. \quad (§ 291)$$

But, $\frac{AB}{A'B'} = \frac{R}{R'} = \frac{r}{r'}$; whence, $\frac{K}{K'} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}$.

That is, *the areas of two similar regular polygons are to each other as the squares of their radii, or as the squares of their apothems.*

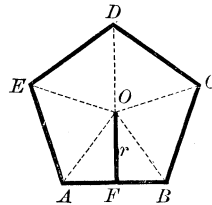
PROP. VI. THEOREM

317. *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*

Given P the perimeter, and r the apothem, of regular polygon AC .

To Prove $\text{area } AC = \frac{1}{2} P \times r$.

($\triangle OAB$, OBC , etc., have common altitude r .)



Ex. 5. One side of a regular hexagon is $2\sqrt{3}$. Find the radius of the inscribed circle.

Ex. 6. The apothem of a regular polygon is 9, and the area 182. What is the perimeter?

Ex. 7. In an equilateral triangle find the area K in terms of the apothem r .

Ex. 8. In a regular hexagon find the apothem in terms of R ; find the area in terms of R ; find the area in terms of the apothem.

Ex. 9. Find the ratio of the altitude and apothem of an equilateral triangle.

Ex. 10. A triangle whose area is 120 is inscribed in a semicircle whose radius is 13. Construct the triangle.

PROP. VII. PROBLEM

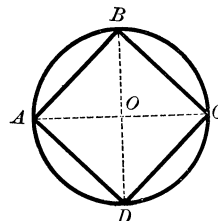
318. *To inscribe a square in a given circle.*

Given $\odot AC$.

Required to inscribe a square in $\odot AC$.

Construction. Draw $\perp$ diameters AC and BD , and lines AB , BC , CD , and DA ; $ABCD$ is a square.

(The proof is left to the pupil; see § 310.)



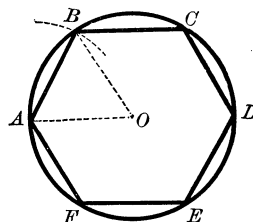
319. Denoting radius OA by R ,

$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 = 2 R^2$ (§ 252); whence, $AB = R\sqrt{2}$.

That is, *the side of an inscribed square is equal to the radius of the circle multiplied by $\sqrt{2}$.*

PROP. VIII. PROBLEM

320. *To inscribe a regular hexagon in a given circle.*



Given $\odot AC$.

Required to inscribe a regular hexagon in $\odot AC$.

Construction. 1. Find point B in circumference at a distance from A equal to radius OA .

2. Line AB is a side of a regular inscribed hexagon; and to inscribe a regular hexagon in a $\odot$ apply radius 6 times as a chord.

Proof. 3. $\triangle AOB$ is equiangular. (§ 51)

4. Then $\angle AOB$ is $\frac{1}{3}$ of 2 rt. $\angle$ (?), or $\frac{1}{6}$ of 4 rt. $\angle$.

321. It follows from § 320 that *the side of a regular inscribed hexagon equals the radius of the circle.*

322. Note. If chords be drawn joining the alternate vertices of a regular inscribed hexagon, there is formed an inscribed equilateral triangle.

323. Let AB be a side of an equilateral $\triangle$ inscribed in $\odot AD$, whose radius is R .

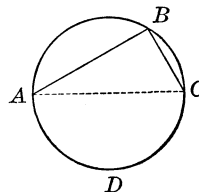
Drawing diameter AC , and chord BC , BC is a side of a regular inscribed hexagon, and therefore equal to R . (§§ 321, 322)

Now ABC is a rt. $\triangle$. (§ 194)

Then by § 253, $\overline{AB}^2 = \overline{AC}^2 - \overline{BC}^2 = (2R)^2 - R^2 = 3R^2$.

Whence,

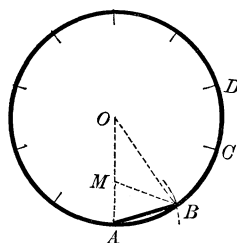
$$AB = R\sqrt{3}.$$



Then, the side of an inscribed equilateral triangle equals the radius of the circle multiplied by $\sqrt{3}$.

PROP. IX. PROBLEM

324. To inscribe a regular decagon in a given circle.



Given $\odot AC$.

Required to inscribe a regular decagon in $\odot AC$.

Construction. 1. Divide radius OA internally in extreme and mean ratio (§ 273), making

$$\frac{OA}{OM} = \frac{OM}{AM}. \quad (1)$$

2. OM is a side of a regular inscribed decagon.

Proof. 3. Take chord $AB = OM$, and draw lines OB , BM ; in $\triangle OAB$, ABM , $\angle A = \angle A$.

4. Since $OM = AB$, (1) becomes $\frac{OA}{AB} = \frac{AB}{AM}$.

5. $\triangle OAB$, ABM are similar (§ 242); then $\angle ABM = \angle O$.

6. $\triangle ABM$ is isosceles, being similar to $\triangle OAB$.

7. By Ax. 1, $AB = BM = OM$; then $\angle OBM = \angle O$. (?)

8. Add results of (5) and (7), $\angle OBA = 2 \angle O$. (2)

9. Find sum of $\angle$ s of $\triangle OAB$; $2 \angle OBA + \angle O = 180^\circ$. (?)

10. Substitute in this value of $\angle OBA$ in (2), then

$$5 \angle O = 180^\circ; \text{ and } \angle O \text{ is } \frac{1}{5} \text{ of } 4 \text{ rt. } \angle.$$

325. Note. If chords be drawn joining the alternate vertices of a regular inscribed decagon, there is formed a regular inscribed pentagon.

326. Denoting the radius of the $\odot$ by R , we have

$$AB = OM = \frac{R(\sqrt{5}-1)}{2}. \quad (\S\ 274)$$

This is an expression for the side of a regular inscribed decagon in terms of the radius of the circle.

Ex. 11. The diameter of a circle is 20. Inscribe a regular hexagon and find its area.

Ex. 12. The side of a regular hexagon is 6; find its area. If this hexagon be an inscribed one, what is the area of the regular hexagon circumscribing the same circle?

Ex. 13. A regular polygon is inscribed in a circle. Give a general method for finding its area.

Ex. 14. An equilateral triangle is inscribed in a circle. If its side is 10, what is its area, and how does it compare with the area of the equilateral triangle circumscribing the same circle?

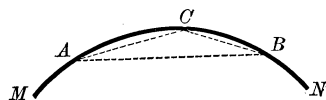
Ex. 15. If the altitude of the triangle in exercise 14 were 10, what would be its area and the area of the equilateral triangle circumscribing the circle?

Ex. 16. The centre of a regular hexagon bisects every line drawn through it which terminates in the sides of the hexagon.

Ex. 17. The apothem of an inscribed equilateral triangle is one-half the radius of the circle.

PROP. X. PROBLEM

327. *To construct the side of a regular pentedecagon inscribed in a given circle.*



Given arc MN .

Required to construct the side of a regular inscribed polygon of fifteen sides.

Construction. If AB is a side of a regular inscribed hexagon (§ 321), and AC a side of a regular inscribed decagon (§ 326), arc BC is $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$ of the circumference.

Ex. 18. An equilateral triangle is inscribed in a circle. The distance from the intersection of the medians to the middle point of the base is 6. Find the radius of the circle.

Ex. 19. If a circle be inscribed in an equilateral triangle, the altitude of the triangle passes through the centre of the circle, and is three times the radius of the circle.

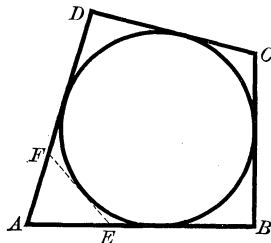
Ex. 20. The area of the square which circumscribes a circle is twice that of the inscribed square.

Ex. 21. The area of the equilateral triangle circumscribing a circle is four times the area of the inscribed one.

MEASUREMENT OF THE CIRCLE

PROP. XI. THEOREM

328. *The circumference of a circle is shorter than the perimeter of any circumscribed polygon.*



Given polygon $ABCD$ circumscribed about a $\odot$.

To Prove circumference of $\odot$ shorter than perimeter $ABCD$.

Proof. 1. Of the perimeters of the $\odot$, and of all possible circumscribed polygons, there must be some perimeter such that all the others are of the same or greater length.

2. But no circumscribed polygon can have this perimeter; for suppose polygon $ABCD$ to have this perimeter, and draw line EF tangent to the $\odot$, meeting AB at E and AD at F .

3. We know that EF is $< (AE + AF)$. (Ax. 6)

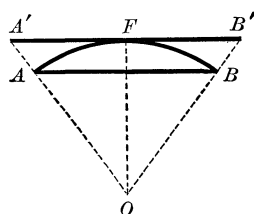
4. Then, the perimeter of polygon $BEFDC$ is $<$ the perimeter of polygon $ABCD$.

5. Hence, the circumference of the $\odot$ is $<$ the perimeter of any circumscribed polygon.

PROP. XII. THEOREM

329. *If a regular polygon be inscribed in, or circumscribed about, a circle, and the number of its sides be indefinitely increased,*

- I. *Its perimeter approaches the circumference as a limit.*
- II. *Its area approaches the area of the circle as a limit.*



Given p , P perimeters, k , K areas, of two regular polygons of same number of sides, respectively inscribed in, and circumscribed about, a $\odot$, whose circumference is C , and area S .

To Prove that, if the number of sides of the polygons be indefinitely increased, P and p approach the limit C , and K and k the limit S .

Proof. 1. Let $A'B'$ be side of polygon whose perimeter is P ; draw radius OF to point of contact.

2. If OA' and OB' cut circumference at A and B , AB is side of polygon whose perimeter is p . (§ 309)

3. Polygons are similar; then $\frac{P}{p} = \frac{OA'}{OF}$. (§§ 314, 315)

4. Then,

$$\frac{P - p}{p} = \frac{OA' - OF}{OF} (?) ; \text{ or } P - p = \frac{p}{OF} \times (OA' - OF).$$

5. p is $<$ circumference; $OA' - OF$ is $<$ $A'F$. (Ax. 7, § 62)

6. Then, $P - p < \frac{C}{OF} \times A'F$. (1)

7. If number of sides of each polygon be indefinitely increased, the polygons continuing to have same number of sides, the length of each side will be indefinitely diminished, and $A'F$ will approach the limit 0.

8. $\frac{C}{OF}$ being constant, $P - p$ will approach limit 0.
9. Now circumference is $< P$, and $> p$. (§ 328, Ax. 7)
10. Then, $P - C$ and $C - p$ will approach limit 0, and P and p will approach limit C .
11. Again, by § 316, $\frac{K}{k} = \frac{\overline{OA'}^2}{\overline{OF}^2}$; or $\frac{K - k}{k} = \frac{\overline{OA'}^2 - \overline{OF}^2}{\overline{OF}^2}$.
(?)
12. Then, $K - k = \frac{k}{\overline{OF}^2} \times \overline{A'F}^2$. (§ 253)
13. If the number of sides of each polygon be indefinitely increased, the polygons continuing to have the same number of sides, $A'F$ will approach limit 0.
14. $\frac{k}{\overline{OF}^2} \times \overline{A'F}^2$ being $< \frac{S}{\overline{OF}^2} \times \overline{A'F}^2$, will approach limit 0.
15. Then, $K - k$ will approach limit 0; and S being $< K$, and $> k$, K and k will approach limit S .

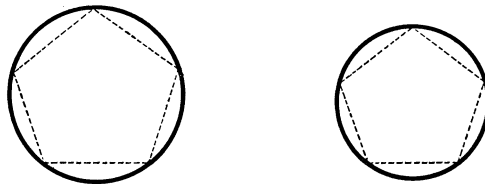
330. *If a regular polygon be inscribed in a circle, and the number of its sides be indefinitely increased, its apothem approaches the radius of the circle as a limit.*

For, it was shown in § 329 that $OA' - OF$ approaches the limit 0, whence OF approaches the limit OA' .

OF is the apothem of a regular polygon inscribed in a circle whose radius is OA' .

PROP. XIII. THEOREM

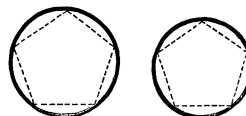
331. *The circumferences of two circles are to each other as their radii.*



Given C and C' the circumferences of two $\odot$ whose radii are R and R' , respectively.

To Prove
$$\frac{C}{C'} = \frac{R}{R'}. \quad (1)$$

Proof. 1. Inscribe similar regular polygons whose perimeters are P and P' , in $\odot$ whose radii are R and R' .



2. Then $\frac{P}{P'} = \frac{R}{R'}$ (§ 315); whence $P \times R' = P' \times R$.

3. If number of sides of polygons be indefinitely increased, $P \times R'$ approaches limit $C \times R'$, and $P' \times R$ the limit $C' \times R$. (§ 329, I)

4. Equate these limits (§ 187), and use § 218.

332. Multiplying the terms of the ratio $\frac{R}{R'}$ by 2, we have

$$\frac{C}{C'} = \frac{2R}{2R'} = \frac{D}{D'},$$

if D and D' denote the diameters of the $\odot$ whose radii are R and R' , respectively.

That is, *the circumferences of two circles are to each other as their diameters.*

333. The proportion $\frac{C}{C'} = \frac{D}{D'}$ (§ 332) may be written

$$\frac{C}{D} = \frac{C'}{D'}. \quad (\S\ 220)$$

That is, *the ratio of the circumference of a circle to its diameter has the same value for every circle.*

This constant value is denoted by the symbol π ; then,

$$\frac{C}{D} = \pi. \quad (1)$$

It is shown by methods of higher mathematics that the ratio π is incommensurable; its numerical value can only be obtained approximately.

Its value to the nearest fourth decimal place is 3.1416.

334. Equation (1) of § 333 gives

$$C = \pi D.$$

That is, *the circumference of a circle is equal to its diameter multiplied by π .*

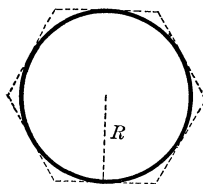
We also have $C = 2 \pi R$.

That is, *the circumference of a circle is equal to its radius multiplied by 2π .*

335. Def. In circles of different radii, *similar arcs, similar segments, and similar sectors* are those which correspond to equal central angles.

PROP. XIV. THEOREM

336. *The area of a circle is equal to one-half the product of its circumference and radius.*



Given R the radius, C the circumference, and S the area of a $\odot$.

To Prove $S = \frac{1}{2} C \times R$. (1)

Proof. 1. Circumscribe a regular polygon about the $\odot$; let P denote its perimeter, and K its area.

2. We have $K = \frac{1}{2} P \times R$. (§ 317)

3. If number of sides of polygon be indefinitely increased, K approaches limit S , and $\frac{1}{2} P \times R$ the limit $\frac{1}{2} C \times R$. (§ 329)

4. Equating limits (?) gives equation (1).

337. We have $C = 2 \pi R$; then, $S = \pi R \times R = \pi R^2$.

That is, *the area of a circle is equal to the square of its radius multiplied by π .*

Again, $S = \frac{1}{4} \pi \times 4 R^2 = \frac{1}{4} \pi \times (2 R)^2$.

If D denote the diameter of the $\odot$, $S = \frac{1}{4} \pi D^2$.

That is, *the area of a circle is equal to the square of its diameter multiplied by $\frac{1}{4} \pi$.*

338. Let S and S' denote the areas of two $\odot$ whose radii are R and R' , and diameters D and D' , respectively.

$$\text{Then,} \quad \frac{S}{S'} = \frac{\pi R^2}{\pi R'^2} = \frac{R^2}{R'^2},$$

$$\text{and} \quad \frac{S}{S'} = \frac{\frac{1}{4}\pi D^2}{\frac{1}{4}\pi D'^2} = \frac{D^2}{D'^2}. \quad (\S\ 337)$$

That is, *the areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.*

339. Let s be the area, and c the arc, of a sector of a $\odot$, whose area is S , circumference C , and radius R .

Since a sector is the same part of the $\odot$ that its arc is of the circumference,

$$\frac{s}{S} = \frac{c}{C}, \text{ or } s = c \times \frac{S}{C}.$$

Then, by $\S\ 336$, $s = c \times \frac{1}{2} R = \frac{1}{2} c \times R$.

Hence, *the area of a sector equals one-half the product of its arc and radius.*

Since similar sectors are like parts of the $\odot$ to which they belong ($\S\ 335$), it follows that

Similar sectors are to each other as the squares of their radii.

Ex. 22. The area of a circle is 64π ; find the circumference.

Ex. 23. Give geometric method for cutting the largest possible octagon from a board one foot square.

Ex. 24. The area of a circle is 154, find its circumference and radius.

Ex. 25. The area of a circle is 144π . Find the area of an inscribed equilateral triangle.

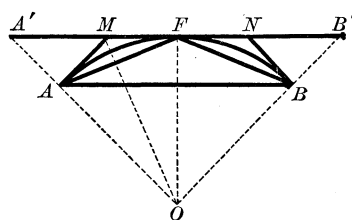
Ex. 26. The radius of a circle is 14; find its area. If the radius were doubled, how would the area be affected?

Ex. 27. What is the area of the largest circle which can be cut from a triangular piece of cardboard whose edges are 5, 13, and 12 inches, respectively?

Ex. 28. The ratio of the areas of two circles is $\frac{9}{16}$, one circumference is 37.6992; find the other circumference. Will more than one circumference satisfy the given conditions? Why? ($\pi = 3.1416$.)

PROP. XV. PROBLEM

340. *Given p and P , the perimeters of a regular inscribed and of a regular circumscribed polygon of the same number of sides, to find p' and P' , the perimeters of a regular inscribed and of a regular circumscribed polygon having double the number of sides.*



Solution. 1. Let AB be side of polygon whose perimeter is p .

2. Let radius OF bisect arc AB at F ; let radii OA and OB extended cut tangent at F at A' and B' , respectively; then $A'B'$ is side of polygon whose perimeter is P . (§ 309)

3. Draw chords AF , BF , and tangents AM , BN , meeting $A'B'$ at M and N , respectively; then AF and MN are sides of polygons whose perimeters are p' and P' . (§ 312)

4. If n denotes number of sides of polygons whose perimeters are p and P , and $2n$ number of sides of polygons whose perimeters are p' and P' ,

$$AB = \frac{p}{n}, A'B' = \frac{P}{n}, AF = \frac{p'}{2n}, MN = \frac{P'}{2n}. \quad (1)$$

5. Line OM bisects $\angle A'OF$; whence, $\frac{A'M}{MF} = \frac{OA'}{OF}$. (§§ 175, 230)

6. Again, $\frac{P}{p} = \frac{OA'}{OF}$. (§ 315)

7. Then, $\frac{P}{p} = \frac{A'M}{MF}$, and $\frac{P+p}{p} = \frac{A'M+MF}{MF}$. (?)

8. Then, $\frac{P+p}{2p} = \frac{A'F}{2MF} = \frac{\frac{1}{2}A'B'}{MN} = \frac{P}{2n} \div \frac{P'}{2n} = \frac{P}{P'}$.

9. Then, $P' \times (P + p) = 2 P \times p$, or $P' = \frac{2 P + p}{P + p}$. (2)

10. Again, since $\angle ABF = \angle AFM$, $\triangle ABF, AFM$ are similar. (§§ 192, 196, 236)

11. Then $\frac{AF}{AB} = \frac{MF}{AF'}$

or

$$\overline{AF}^2 = AB \times MF. \quad (?)$$

12. Then by (1), $\frac{p'^2}{4n^2} = \frac{p}{n} \times \frac{P'}{4n} = \frac{p \times P'}{4n^2}$.

13. Then, $p'^2 = p \times P'$, and $p' = \sqrt{p \times P'}$. (3)

341. We will now show how to compute an approximate value of π (§ 333).

If the diameter of a $\odot$ is 1, the side of an inscribed square is $\frac{1}{2}\sqrt{2}$ (§ 319); hence, its perimeter is $2\sqrt{2}$.

Again, the side of a circumscribed square is equal to the diameter of the $\odot$; hence, its perimeter is 4.

We then put in equation (2), § 340,

$$P = 4, \text{ and } p = 2\sqrt{2} = 2.82843.$$

Then, $P' = \frac{2 P \times p}{P + p} = 3.31371$.

We then put in equation (3), § 340,

$$p = 2.82843, \text{ and } P' = 3.31371.$$

Then, $p' = \sqrt{p \times P'} = 3.06147$.

These are the perimeters of the regular circumscribed and inscribed octagons, respectively.

Repeating the operation with these values, we put in (2),

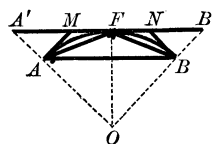
$$P = 3.31371, \text{ and } p = 3.06147.$$

Then, $P' = \frac{2 P \times p}{P + p} = 3.18260$.

We then put in (3), $p = 3.06147$ and $P' = 3.18260$.

Then, $p' = \sqrt{p \times P'} = 3.12145$.

These are, respectively, the perimeters of the regular circumscribed and inscribed polygons of sixteen sides.



In this way, we form the following table:

NO. OF SIDES	PERIMETER OF REG. CIRC. POLYGON	PERIMETER OF REG. INSC. POLYGON
4	4.	2.82843
8	3.31371	3.06147
16	3.18260	3.12145
32	3.15172	3.13655
64	3.14412	3.14033
128	3.14222	3.14128
256	3.14175	3.14151
512	3.14163	3.14157

The last result shows that the circumference of a $\odot$ whose diameter is 1 is > 3.14157 , and < 3.14163 .

Hence, an approximate value of π is 3.1416, correct to the fourth decimal place.

Note. The value of π to fourteen decimal places is 3.14159265358979.

Ex. 29. The radius of a circle is 12. What is the radius of a circle having twice the area?

Ex. 30. The area of the equilateral triangle inscribed in a circle is one-half the area of the regular hexagon inscribed in the same circle.

Ex. 31. Two circumferences are in the ratio 3 to 4. What is the ratio of their radii? their diameters? their areas?

Ex. 32. The areas of two circles are in the ratio 1 to 2. What is the ratio of their radii? of their diameters?

Ex. 33. The areas of two circles are 836 and 616, respectively. If the radius of the second is 257, what is the radius of the first?

Ex. 34. One side of an inscribed equilateral triangle is 10; find the radius of the circle.

Ex. 35. What is the area of a sector of a circle whose radius is 12, if the angle of the sector is 45° ?

Ex. 36. Find the ratio of the area of a circle to the area of its circumscribed square.

Ex. 37. Find the ratio of the area of a circle to the area of its circumscribed equilateral triangle.

Ex. 38. Find the ratio of the perimeters of the circumscribed and inscribed equilateral triangles.

Ex. 39. A wheel revolves 55 times in travelling $\frac{1045\pi}{4}$ ft. What is its diameter in inches?

If r represents the radius, a the apothem, s the side, and k the area, prove that

Ex. 40. In a regular octagon,

$$s = r\sqrt{2 - \sqrt{2}}, a = \frac{1}{2}r\sqrt{2 + \sqrt{2}}, \text{ and } k = 2r^2\sqrt{2}.$$

Ex. 41. In a regular dodecagon,

$$s = r\sqrt{2 - \sqrt{3}}, a = \frac{1}{2}r\sqrt{2 + \sqrt{3}}, \text{ and } k = 3r^2.$$

Ex. 42. In a regular octagon,

$$s = 2a(\sqrt{2} - 1), r = a\sqrt{4 - 2\sqrt{2}}, \text{ and } k = 8a^2(\sqrt{2} - 1).$$

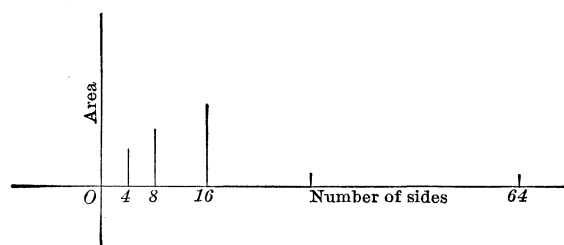
Ex. 43. In a regular dodecagon,

$$s = 2a(2 - \sqrt{3}), r = 2a\sqrt{2 - \sqrt{3}}, \text{ and } k = 12a^2(2 - \sqrt{3}).$$

Ex. 44. In a regular decagon, $a = \frac{1}{4}r\sqrt{10 + 2\sqrt{5}}$.
(Find the apothem.)

Ex. 45. Find the number of degrees in a radian, an arc whose length is equal to that of the radius of the circle. ($\pi = 3.1416$.)

Ex. 46. A circular grass plot whose diameter is 52 feet is surrounded by a walk of uniform width whose area is 753.984. Find the width of the walk. ($\pi = 3.1416$.)



Ex. 47. Find the areas of regular inscribed polygons of 4, 8, 16, 32, 64, 128 sides, and of the circumscribing circle (§ 341). Using *area* for vertical measurements and *number of sides* for horizontal measurements, make a graph of the data showing the approach of the area of the polygon to that of the circle as the number of sides increases.

LOCI

342. Def. If a series of points, all of which satisfy a certain condition, lie in a certain line, and every point in this line satisfies the given condition, the line is said to be the *locus* of the points.

For example, all points which satisfy the condition of being equally distant from the extremities of a straight line, lie in the perpendicular erected at the middle point of the line (§ 56).

Also, every point in the perpendicular erected at the middle point of a line satisfies the condition of being equally distant from the extremities of the line (§ 55).

Hence, *the perpendicular erected at the middle point of a straight line is the Locus of points which are equally distant from the extremities of the line.*

Again, all points which satisfy the condition of being within an angle, and equally distant from its sides, lie in the bisector of the angle (§ 99).

Also, every point in the bisector of an angle satisfies the condition of being equally distant from its sides (§ 98).

Hence, *the bisector of an angle is the locus of points which are within the angle, and equally distant from its sides.*

Ex. 48. Find the locus of points in a plane three inches from a fixed point in the plane. Construct the locus.

Ex. 49. What is the locus of points in a plane equally distant from all points in the circumference of a circle?

Ex. 50. Find the locus of the vertices of triangles having a fixed base and the same altitude. Construct the locus.

Ex. 51. Two equal chords intersect; find the locus of their intersections.

Ex. 52. The length of the hypotenuse of a right triangle is fixed; find the locus of the centres of the inscribed circles; of the circumscribed circles.

Ex. 53. Find the locus of the middle points of chords drawn through a fixed point within a circle.

Ex. 54. Construct the locus of points equidistant from two given lines. How many cases are there?

Ex. 55. Draw the approximate locus of the middle points of the hypotenuses of all right triangles having a hypotenuse of fixed length and the vertex of the right angle fixed.

Ex. 56. Construct the locus of the centres of circles tangent to two lines which intersect at right angles. Is there more than one system of these circles?

Ex. 57. Find the locus of the centres of circumferences which pass through a fixed point.

NEW SOLID GEOMETRY

SOLID GEOMETRY

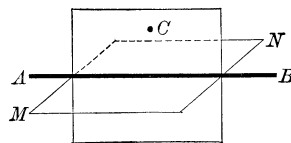
BOOK VI

LINES AND PLANES IN SPACE. DIEDRAL ANGLES. POLYEDRAL ANGLES

343. Def. A plane is said to be *determined* by certain lines or points when one plane, and only one, can be drawn through these lines or points.

PROP. I. THEOREM

344. *A plane is determined by a straight line and a point without the line.*



Given point C without str. line AB .

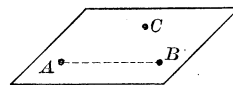
To Prove that a plane is determined by AB and C .

Proof. 1. If any plane MN be drawn through AB , it may be revolved about AB as an axis until it contains point C .

2. Hence, a plane can be drawn through AB and C ; and it is evident that but one such plane can be drawn.

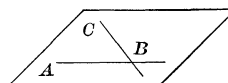
345. *A plane is determined by three points not in the same straight line.*

For if two of the points be joined by a straight line, one plane, and but one, can be drawn through this line and the remaining point (§ 344).



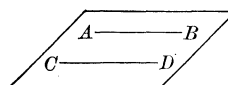
346. *A plane is determined by two intersecting straight lines.*

For, by § 344, one plane, and but one, can be drawn through AB and C ; and this plane contains BC by § 6.



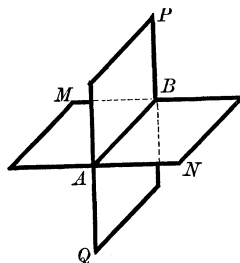
347. *A plane is determined by two parallel straight lines.*

By the definition of § 68, two $\parallel$ s lie in the same plane; and by § 344, but one plane can be drawn through A and any point C of CD .



PROP. II. THEOREM

348. *The intersection of two planes is a straight line.*



Given line AB the intersection of planes MN and PQ .

To Prove AB a str. line.

Proof. 1. Draw a str. line between points A and B .

2. By § 6, this str. line lies in plane MN , and also in plane PQ .

3. Then it must be the intersection of planes MN and PQ .

4. Hence, the line of intersection AB is a str. line.

Ex. 1. Can a triangle be a warped surface, that is, must all parts of it lie in the same plane? Why? Is this true of any other rectilinear figure?

Ex. 2. How many planes determine a line? How many planes are *sufficient* to determine a point? A solid? Is the number of points necessary to determine these bounding planes the same as the number of planes?

Ex. 3. Fold a sheet of paper. Is the fold a curved line? A broken line? A straight line? Why?

349. Defs. If a straight line meets a plane, the point of intersection is called the *foot* of the line.

A straight line is said to be *perpendicular to a plane* when it is perpendicular to every straight line drawn in the plane through its foot.

A straight line is said to be *parallel to a plane* when it cannot meet the plane however far they may be extended.

A straight line which is neither perpendicular nor parallel to a plane, is said to be *oblique* to it.

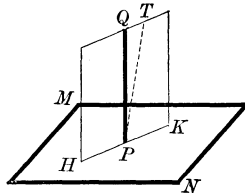
Two planes are said to be *parallel to each other* when they cannot meet however far they may be extended.

350. The following is given for reference :

A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.

PROP. III. THEOREM

351. *At a given point in a plane, but one perpendicular to the plane can be drawn.*



Given line $PQ \perp$ to plane MN at P .

To Prove PQ the only $\perp$ which can be drawn to MN at P .

Proof. 1. If possible, let PT be another $\perp$ to MN at P .

2. Let the plane determined by PQ and PT intersect MN in line HK ; then, both PQ and PT are $\perp$ HK .

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 350)

3. But this is impossible; for, in plane HKT , only one $\perp$ can be drawn to HK at P .

[At a given point in a str. line, but one $\perp$ to the line can be drawn.]

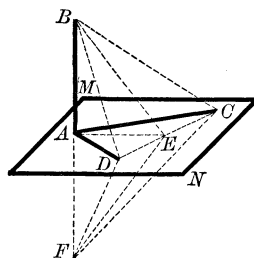
(§ 15)

4. Then only one $\perp$ can be drawn to MN at P .

Note. As a rigorous proof of the theorem that, at a given point in a plane, a perpendicular to the plane can be drawn, is too difficult for the average pupil, it is omitted.

PROP. IV. THEOREM

352. *A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to their plane.*



Given line $AB \perp$ lines AC and AD , and plane MN drawn through AC and AD .

To Prove $AB \perp$ plane MN .

Proof. 1. Let AE be any other str. line drawn through A in MN ; draw line CD intersecting AC at C , AE at E , AD at D .

2. Prolong BA to F , making $AF = AB$, and draw lines BC , BE , BD , FC , FE , and FD .

3. In $\triangle BCD$ and CDF , $CD = CD$.

4. Since AC and AD are $\perp$ to BF at its middle point,

$$BC = CF, \text{ and } BD = DF.$$

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 54, I)

5. Then, $\triangle BCD = \triangle CDF$.

[Two $\triangle$ are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 52)

6. Now revolve $\triangle BCD$ about CD as an axis until it coincides with $\triangle CDF$.

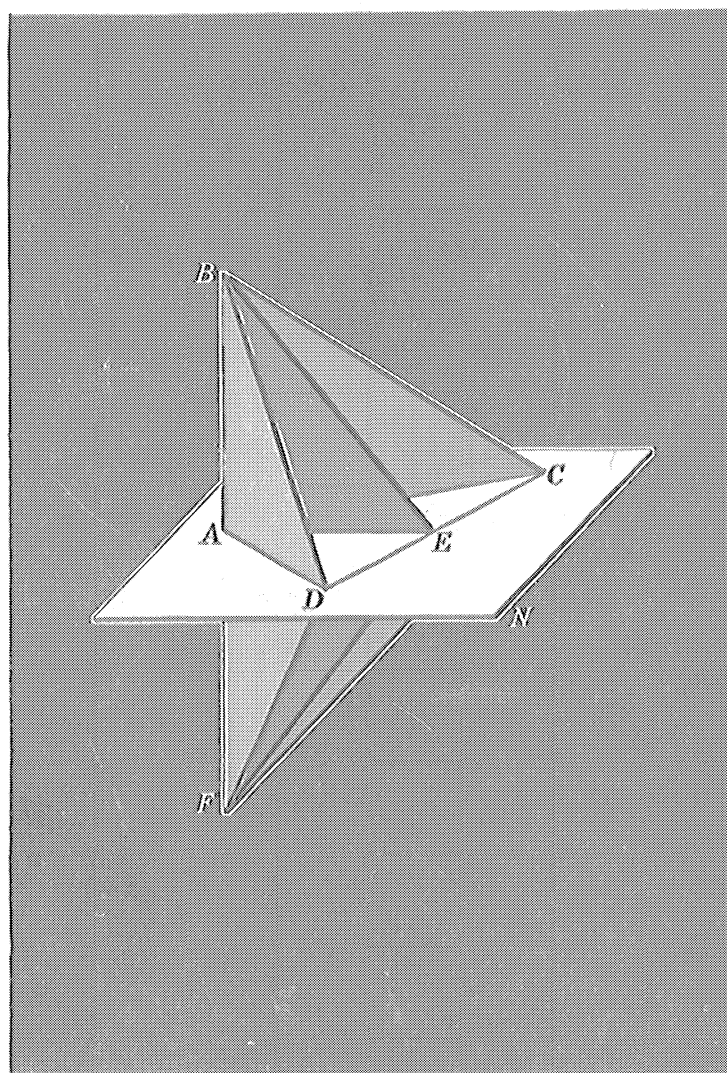


PLATE VII

7. Then, point B will fall on point F , and line BE will coincide with line EF ; that is, $BE = EF$.

8. Hence, since points A and E are each equally distant from B and F , line AE is $\perp BF$.

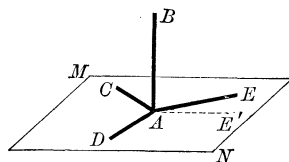
[Two points, each equally distant from the extremities of a str. line, determine a $\perp$ at its middle point.] (§ 56)

9. Since AE is *any* str. line, AB is $\perp$ to *every* str. line drawn through A in MN , and is therefore $\perp$ to plane MN .

[A str. line is said to be $\perp$ to a plane when it is $\perp$ to every str. line drawn in the plane through its foot.] (§ 349)

PROP. V. THEOREM

353. *All the perpendiculars to a straight line at a given point lie in a plane perpendicular to the line.*



Given AC , AD , and AE any three $\perp$ s to line AB at A .

To Prove that they lie in a plane $\perp$ to AB .

Proof. 1. Let MN be the plane determined by AC and AD ; then, plane MN is $\perp AB$.

[A str. line $\perp$ to each of two str. lines at their point of intersection is $\perp$ to their plane.] (§ 352)

2. Let the plane determined by AB and AE intersect MN in line AE' ; then, $AB \perp AE'$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 350)

3. In plane ABE , only one $\perp$ can be drawn to AB at A .

[At a given point in a str. line, but one $\perp$ to the line can be drawn.] (§ 15)

4. Then, AE' and AE coincide, and AE lies in plane MN .

5. Then, all the $\perp$ s to AB at A lie in a plane $\perp AB$.

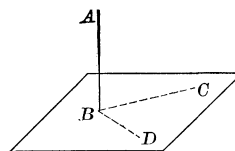
354. It follows from § 353 that

Through a given point in a straight line, a plane can be drawn perpendicular to the line, and but one.

355. *Through a given point without a straight line, a plane can be drawn perpendicular to the line, and but one.*

If C is the given point without line AB , draw line $CB \perp AB$, and BD any other $\perp$ to AB at B .

Then, the plane drawn through BC and BD will be $\perp AB$.



[A str. line $\perp$ to each of two str. lines at their point of intersection is $\perp$ to their plane.] (§ 352)

Again, every plane through $C \perp AB$ must intersect the plane determined by AB and BC in a line from $C \perp AB$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 350)

But only one $\perp$ can be drawn from C to AB .

[From a given point without a str. line, but one $\perp$ can be drawn to the line.] (§ 59)

Then, every plane through $C \perp AB$ must contain BC , and be $\perp$ to AB at B .

But only one plane can be drawn through $B \perp AB$.

[Through a given point in a str. line, but one plane can be drawn $\perp$ to the line.] (§ 354)

Hence, but one plane can be drawn through $C \perp AB$.

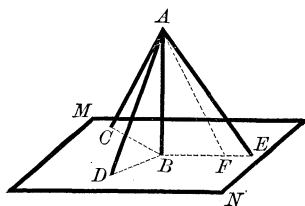
Ex. 4. If we have five points, four of which are in same plane, how many planes do they determine? If no four of the five were in the same plane, would the number of planes change?

PROP. VI. THEOREM

356. *If from a point in a perpendicular to a plane, oblique lines be drawn to the plane,*

I. *Two oblique lines cutting off equal distances from the foot of the perpendicular are equal.*

II. *Of two oblique lines cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.*



Given line $AB \perp$ plane MN at B ; AC and AD oblique lines meeting MN at equal distances from B , and AE an oblique line meeting MN at a greater distance from B than AC .

To Prove $AC = AD$, and $AE > AC$.

Proof. 1. Draw lines BC , BD , and BE .

2. In $\triangle ABC$, ABD , $AB = AB$; and by hyp., $BC = BD$.

3. Also, $\angle ABC = \angle ABD$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 350)

4. Then, $\triangle ABC = \triangle ABD$.

[Two $\triangle$ are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 46)

5. Then, $AC = AD$.

[In equal figures, the homologous parts are equal.] (§ 48)

6. Again, on BE take $BF = BC$, and draw line AF ; then, since $BF = BC$,

$$AF = AC.$$

[If from a point in a $\perp$ to a plane, oblique lines be drawn to the plane, two oblique lines cutting off equal distances from the foot of the $\perp$ are equal.] (§ 356, I)

7. But, $AB \perp BE$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 350)

8. Then, $AE > AF$, or AC .

[If oblique lines be drawn from a point to a str. line, of two oblique lines cutting off unequal distances from the foot of the $\perp$ from the point to the line, the more remote is the greater.] (§ 64, II)

PROP. VII. THEOREM

357. (Converse of Prop. VI.) *If from a point in a perpendicular to a plane, oblique lines be drawn to the plane,*

I. *Two equal oblique lines cut off equal distances from the foot of the perpendicular.*

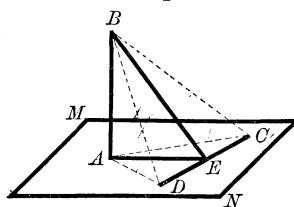
II. *Of two unequal oblique lines, the greater cuts off the greater distance from the foot of the perpendicular.*

Given line $AB \perp$ to plane MN at B ; AC , AD , and AE oblique lines from A to MN , AC and AD being equal and $AE > AC$; also, lines BC , BD , and BE . (Fig. of Prop. VI.)

To Prove $BC = BD$, and $BE > BC$. (The proof is left to the pupil.)

PROP. VIII. THEOREM

358. *If through the foot of a perpendicular to a plane a line be drawn at right angles to any line in the plane, the line drawn from its intersection with this line to any point in the perpendicular will be perpendicular to the line in the plane.*



Given line $AB \perp$ to plane MN at A , line $AE \perp$ to any line CD in MN , and line BE from E to any point B in AB .

To Prove $BE \perp CD$.

Proof. 1. On CD take $EC = ED$.

2. Draw lines AC , AD , BC , and BD ; then, $AC = AD$.

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 54, I)

3. Then, $BC = BD$.

[If from a point in a $\perp$ to a plane, oblique lines be drawn to the plane, two oblique lines cutting off equal distances from the foot of the $\perp$ are equal.] (§ 356, I)

360. If AB is the $\perp$, and AC any other line, from A to MN (Fig. of Prop. IX), and line BC be drawn, $AB \perp BC$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 350)

Then, $AB < AC$.

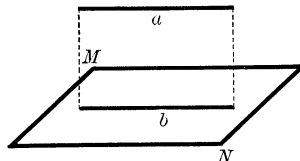
[The $\perp$ is the shortest line that can be drawn from a point to a str. line.] (§ 95)

Then, *the perpendicular is the shortest line that can be drawn from a point to a plane.*

Note. The *distance* of a point from a plane signifies the length of the perpendicular from the point to the plane.

PROP. X. THEOREM

361. *If two straight lines are parallel, a plane drawn through one of them, not coinciding with the plane of the parallels, is parallel to the other.*



Given line $a \parallel$ to line b , and plane MN drawn through b , not coinciding with the plane of the $\parallel$ s.

To Prove $a \parallel MN$.

Proof. 1. Since a and b lie in a plane which intersects MN in b , if a meets MN , it must be somewhere in b .

2. But since a is $\parallel b$, it cannot meet it (§ 68).

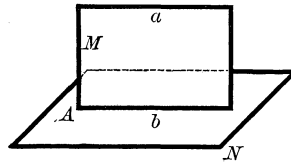
3. Then, a and MN cannot meet, and are $\parallel$ (§ 349).

Ex. 5. If the circumference of a circle be drawn in a plane and at its centre a perpendicular to the plane be erected, any point in this perpendicular is equidistant from the circumference.

Ex. 6. The plane which bisects a straight line at right angles is the locus of all points equidistant from the extremities of the line.

PROP. XI. THEOREM

362. *If a straight line is parallel to a plane, the intersection of the plane with any plane drawn through the line is parallel to the line.*



Given line $a \parallel$ to plane MN ; and b the intersection of MN with any plane drawn through a .

To Prove $a \parallel b$.

(a and b lie in the same plane, and cannot meet.)

363. *If a line and plane are parallel, a parallel to the line through any point of the plane lies in the plane.*

Let line a be $\parallel$ to plane MN (Fig. of Prop. XI), and line b drawn $\parallel$ to a through any point A in MN .

The plane drawn through a and A intersects MN in a $\parallel$ to a .

[If a str. line is $\parallel$ to a plane, the intersection of the plane with any plane drawn through the line is $\parallel$ to the line.] (§ 362)

But through A only one $\parallel$ can be drawn to a .

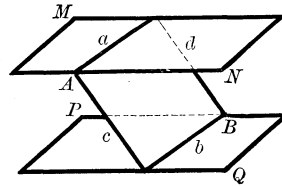
[But one str. line can be drawn through a given point $\parallel$ to a given str. line.]

Whence, b lies in MN .

(§ 69)

PROP. XII. THEOREM

364. *If two parallel planes are cut by a third plane, the intersections are parallel.*



Given $\parallel$ planes MN and PQ cut by plane AB in lines a and b , respectively. **To Prove** $a \parallel b$.

(a and b lie in the same plane, and cannot meet.)

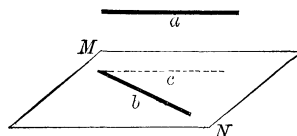
365. If c and d are $\parallel$ lines included between $\parallel$ planes MN and PQ (Fig. of Prop. XII), the figure bounded by a , c , b , and d is a $\square$, and $c = d$.

[In any $\square$, the opposite sides are equal.] (§ 104)

Then, *parallel lines included between parallel planes are equal.*

PROP. XIII. THEOREM

366. *Through a given straight line, a plane can be drawn parallel to any other straight line.*



Given lines a and b .

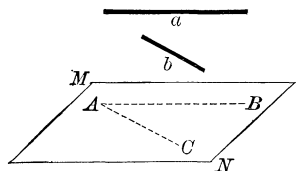
To Prove that a plane can be drawn through $b \parallel a$.

(Draw line $c \parallel a$; then use § 361.)

Note. If a is $\parallel b$, an indefinitely great number of planes can be drawn through $b \parallel a$ (§ 361); otherwise, but one such plane can be drawn, for every plane through $b \parallel a$ must contain c (§ 363), and but one plane can be drawn through b and c .

PROP. XIV. THEOREM

367. *Through a given point a plane can be drawn parallel to any two straight lines in space.*



Given point A and lines a and b .

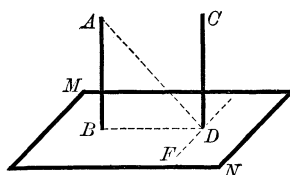
To Prove that a plane can be drawn through $A \parallel$ to a and b .

(The proof is left to the pupil; see § 361.)

Note. If a and b are $\parallel$, an indefinitely great number of planes can be drawn through $A \parallel$ to a and b (§ 361); otherwise, but one such plane can be drawn.

PROP. XV. THEOREM

368. *Two perpendiculars to the same plane are parallel.*



Given lines AB and $CD \perp$ to plane MN at B and D , respectively.

To Prove $AB \parallel CD$.

Proof. 1. Let A be any point of AB , and draw line AD .

2. Also, draw line BD , and line DF in plane $MN \perp BD$.

3. Then, $CD \perp DF$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 350)

4. Also, $AD \perp DF$.

[If through the foot of a $\perp$ to a plane a line be drawn at rt. $\angle$ to any line in the plane, the line drawn from its intersection with this line to any point in the $\perp$ will be $\perp$ to the line in the plane.] (§ 358)

5. Then, CD , AD , and BD lie in the same plane.

[All the $\perp$ to a str. line at a given point lie in a plane $\perp$ to the line.] (§ 353)

6. Since points A and B lie in the plane of the lines AD , BD , and CD , AB lies in this plane.

[A plane is a surface such that the str. line joining any two of its points lies entirely in the surface.] (§ 6)

7. That is, AB and CD lie in the same plane.

8. Again, AB and CD are $\perp BD$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 350)

9. Then, $AB \parallel CD$.

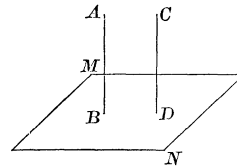
[Two $\perp$ to the same str. line are $\parallel$.] (§ 70)

369. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*

Let lines AB and CD be $\parallel$, and $AB \perp$ to plane MN .

Since two $\perp$ s to the same plane are $\parallel$ (§ 368), a $\perp$ from C to MN is $\parallel AB$.

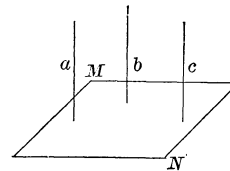
But through C , only one $\parallel$ can be drawn to line AB (§ 69), whence $CD \perp MN$.



370. *If each of two straight lines is parallel to a third straight line, they are parallel to each other.*

Let lines a and b be $\parallel$ line c .

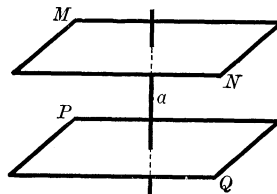
Drawing plane $MN \perp c$, it is $\perp$ to a and b (§ 369); then, $a \parallel b$ (§ 368).



Ex. 7. Find the locus of points three inches from a given plane.

PROP. XVI. THEOREM

371. *Two planes perpendicular to the same straight line are parallel.*



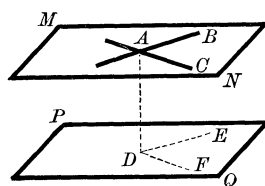
Given planes MN and $PQ \perp$ to line a .

To Prove $MN \parallel PQ$.

(Prove as in § 70; by § 355, but one plane can be drawn through a given point $\perp$ to a given str. line.)

PROP. XVII. THEOREM

372. *If each of two intersecting lines is parallel to a plane, their plane is parallel to the given plane.*



Given lines AB and AC , in plane MN , $\parallel$ to plane PQ .

To Prove $MN \parallel PQ$.

Proof. 1. Draw line $AD \perp PQ$, line $DE \parallel AB$, and line $DF \parallel AC$; then, DE and DF lie in PQ .

[If a line and a plane are $\parallel$, a $\parallel$ to the line through any point of the plane lies in the plane.] (§ 363)

2. Whence, AD is $\perp$ to DE and DF .

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 350)

3. Therefore, AD is $\perp$ to AB and AC .

[A str. line $\perp$ to one of two $\parallel$ s is $\perp$ to the other.] (§ 72)

4. Then, $AD \perp MN$.

[A str. line $\perp$ to each of two str. lines at their point of intersection is $\perp$ to their plane.] (§ 352)

5. Then, $MN \parallel PQ$.

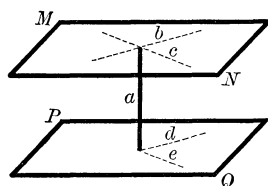
[Two planes $\perp$ to the same str. line are $\parallel$.] (§ 371)

Ex. 8. Find locus of points equidistant from all points in the circumference of a circle.

Ex. 9. A set of points are always equidistant from the vertices of a triangle; find their locus.

PROP. XVIII. THEOREM

373. *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*



Given planes MN and $PQ \parallel$, and line $a \perp PQ$.

To Prove $a \perp MN$.

Proof. 1. Let two planes through a intersect MN in lines b and c , and PQ in lines d and e , respectively.

2. Then, $b \parallel d$, and $c \parallel e$.

[If two $\parallel$ planes are cut by a third plane, the intersections are $\parallel$.]
(§ 364)

3. But a is $\perp$ to d and e .

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.]
(§ 350)

4. Then, a is $\perp$ to b and c .

[A str. line $\perp$ to one of two $\parallel$ s is $\perp$ to the other.]
(§ 72)

5. Then, $a \perp MN$.

[A str. line $\perp$ to each of two str. lines at their point of intersection is $\perp$ to their plane.]
(§ 352)

374. *Two parallel planes are everywhere equally distant.*
(Note, p. 184.)

If planes MN and PQ (Fig. of Prop. XVIII) are $\parallel$, all lines from points in $MN \perp$ to PQ are $\parallel$.

[Two $\perp$ to the same plane are $\parallel$.]
(§ 368)

Therefore, these lines are all equal.

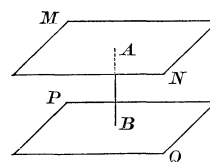
[$\parallel$ lines included between $\parallel$ planes are equal.]
(§ 365)

375. *Through a given point a plane can be drawn parallel to a given plane, and but one.*

Let A be any point without plane PQ ; draw line $AB \perp PQ$, and through A draw plane $MN \perp AB$; then $MN \parallel PQ$.

[Two planes $\perp$ to the same str. line are $\parallel$.]

(§ 371)



If another plane could be drawn through $A \parallel PQ$, it would have to be $\perp AB$.

[A str. line $\perp$ to one of two $\parallel$ planes is $\perp$ to the other also.] (§ 373)

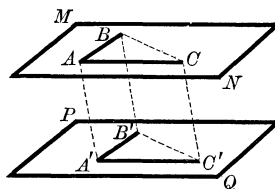
It would then coincide with MN .

[Through a given point in a str. line, but one plane can be drawn $\perp$ to the line.] (§ 354)

Then but one plane can be drawn through $A \parallel PQ$.

PROP. XIX. THEOREM

376. *If two angles not in the same plane have their sides parallel and extending in the same direction, they are equal, and their planes are parallel.*



Given $\angle BAC$ and $B'A'C'$ in planes MN and PQ , respectively, with AB and $A'B'$ $\parallel$, respectively, to AC and $A'C'$, and extending in the same direction.

To Prove $\angle BAC = \angle B'A'C'$, and $MN \parallel PQ$.

Proof. 1. Lay off $AB = A'B'$ and $AC = A'C'$; draw lines AA' , BB' , CC' , BC , and $B'C'$.

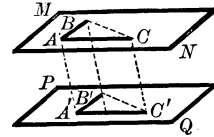
2. Since AB is equal and $\parallel$ to $A'B'$, $ABB'A'$ is a $\square$.

[If two sides of a quadrilateral are equal and $\parallel$, the figure is a $\square$.] (§ 107)

3. Whence, AA' is equal and $\parallel$ to BB' .

[The opposite sides of a $\square$ are equal.]

(§ 104)



4. Similarly, $ACC'A'$ is a $\square$, and AA' is equal and $\parallel$ to CC' .

5. Then, BB' is equal and $\parallel$ to CC' .

[If each of two str. lines is $\parallel$ to a third str. line, they are $\parallel$ to each other.] (§ 370)

6. Whence, $BB'C'C$ is a $\square$, and $BC = B'C'$.

7. Then, $\triangle ABC = \triangle A'B'C'$.

[Two $\triangle$ s are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 52)

8. Then, $\angle BAC = \angle B'A'C'$.

[In equal figures, the homologous parts are equal.] (§ 48)

9. Again, lines AB and AC are $\parallel$ to plane PQ .

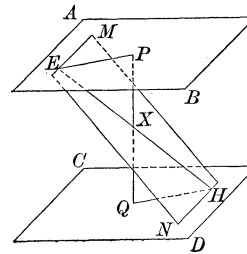
[If two str. lines are $\parallel$, a plane drawn through one of them, not coinciding with the plane of the $\parallel$ s, is $\parallel$ to the other.] (§ 361)

10. Then, $MN \parallel PQ$.

[If each of two intersecting lines is $\parallel$ to a plane, their plane is $\parallel$ to the given plane.] (§ 372)

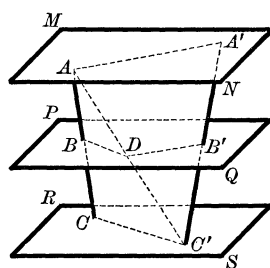
Ex. 10. Are two straight lines which are perpendicular to the same straight line always parallel? Prove your conclusion.

Ex. 11. If a straight line be perpendicular to each of two parallel planes and through its middle point X a plane be drawn intersecting the two parallel planes, then any line drawn in this plane through X and terminating in the parallel planes is bisected at X .



PROP. XX. THEOREM

377. *If two straight lines are cut by three parallel planes, the corresponding segments are proportional.*



Given $\parallel$ planes MN , PQ , and RS intersecting lines AC and $A'C'$ in points A , B , C , and A' , B' , C' , respectively.

To Prove $\frac{AB}{BC} = \frac{A'B'}{B'C'}$.

Proof. 1. Draw line AC' ; let plane through AC and AC' intersect PQ in line BD , and RS in line CC' ; then $BD \parallel CC'$.

[If two $\parallel$ planes are cut by a third plane, the intersections are $\parallel$.]

(§ 364)

2. Then, $\frac{AB}{BC} = \frac{AD}{DC'}$. (1)

[A $\parallel$ to one side of a $\triangle$ divides the other two sides proportionally.]

(§ 227)

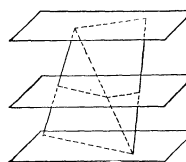
3. In like manner, $\frac{AD}{DC'} = \frac{A'B'}{B'C'}$. (2)

4. From (1) and (2), $\frac{AB}{BC} = \frac{A'B'}{B'C'}$.

[Things which are equal to the same thing are equal to each other.]

(Ax. 1)

Ex. 12. If two lines which have their extremities in two parallel planes be cut by a third plane parallel to these planes, then one of these lines is to each part cut off as the other line is to the corresponding part.



Ex. 13. If two planes are parallel to the same plane, they are parallel to each other.

Ex. 14. Three parallel planes are so placed that the ratio of the distance from the first plane to the second to the distance from the second to the third is $\frac{3}{4}$. A line makes an angle of 45° with one of these planes, and the distance between the first and second plane is 20; find the segments of the line made by these planes.

Ex. 15. A line parallel to each of two intersecting planes is parallel to their intersection.

DIEDRAL ANGLES

DEFINITIONS

378. If two planes meet in a straight line, the figure formed is called a *diedral angle*.

The line of intersection of the planes is called the *edge* of the diedral angle, and the planes the *faces*.

Thus, in the diedral angle formed by planes BD , BF , BE is the edge, and BD , BF the faces.

A diedral angle may be designated by two letters on its edge; or, if several diedral angles have a common edge, by four letters, one in each face and two on the edge, the letters on the edge being named between the other two.

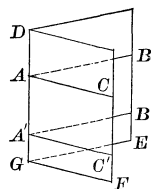
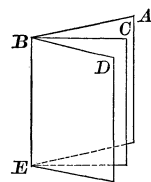
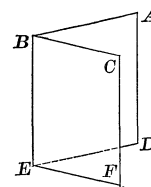
Thus, we may read the above diedral angle BE , or $ABEF$.

We call two diedral angles *adjacent* when they have the same edge, and a common face between them; as, $ABEC$ and $CBED$.

We call two diedral angles *vertical* when the faces of one are the extensions of the faces of the other.

379. A *plane angle* of a diedral angle is the angle between two straight lines drawn one in each face, perpendicular to the edge at the same point.

Thus, if lines AB and AC be drawn in faces DE and DF , respectively, of diedral angle DG , perpendicular to DG at A , $\angle BAC$ is a plane angle of the diedral angle.



380. If BAC and $B'A'C'$ (Fig. of § 379) are plane $\angle$ s of diedral $\angle DG$, $AB \parallel A'B'$ and $AC \parallel A'C'$. (§ 70)

Then, $\angle BAC = \angle B'A'C'$. (§ 376)

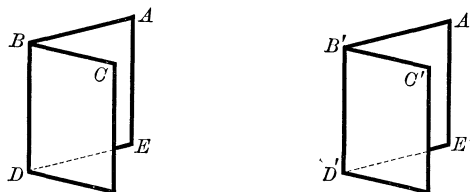
That is, *all plane angles of a diedral angle are equal.*

381. Since a plane perpendicular to the edge of a diedral angle intersects the faces in lines perpendicular to the edge (§ 350), *a plane perpendicular to the edge of a diedral angle intersects the faces in lines which form the plane angle of the diedral angle* (§ 379).

382. Two diedral angles are *equal* when their faces may be made to coincide.

PROP. XXI. THEOREM

383. *Two diedral angles are equal if their plane angles are equal.*



Given ABC and $A'B'C'$ plane $\angle$ s of diedral $\angle BD$ and $B'D'$, respectively, and $\angle ABC = \angle A'B'C'$.

To Prove diedral $\angle BD =$ diedral $\angle B'D'$.

Proof. 1. Place diedrals so that $A'B'$ shall coincide with AB , and $B'C'$ with BC .

2. Then, $BD \perp$ plane ABC , $B'D' \perp$ plane $A'B'C'$. (§ 352)

3. Then, $B'D'$ and BD coincide (§ 351); whence faces of diedrals coincide, and diedrals are equal. (§ 346)

384. It follows from § 383 that

If two planes intersect, their vertical diedral angles are equal.

For their plane $\angle$ s are equal. (§ 44)

PROP. XXII. THEOREM

385. *If two dihedral angles are equal, their plane angles are equal. (Converse of Prop. XXI.)*

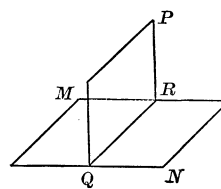
Given ABC and $A'B'C'$ plane $\angle$ s of dihedral $\angle$ s BD and $B'D'$, respectively, and $BD = B'D'$. (Fig. of Prop. XXI.)

To Prove $\angle ABC = \angle A'B'C'$.

(Apply $B'D'$ to BD so that face $A'D'$ shall coincide with AD , and $C'D'$ with CD , point B' falling at B .)

386. Defs. If a plane meets another in such a way as to make the adjacent dihedral angles equal, each is called a *right dihedral angle*, and the planes are said to be *perpendicular*.

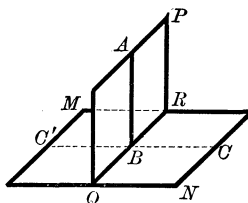
Thus, if plane PQ meets plane MN in such a way as to make dihedral $\angle$ s $PRQM$ and $PRQN$ equal, each of these is a right dihedral $\angle$, and $MN \perp PQ$.



It may be shown as in § 15 that *through a given line in a plane, a plane can be drawn perpendicular to the given plane, and but one.*

PROP. XXIII. THEOREM

387. *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other.*



Given planes PQ and $MN \perp$, intersecting in line QR , and line AB in plane $PQ \perp QR$.

To Prove $AB \perp MN$.

Proof. 1. Draw line $C'B'C$ in plane $MN \perp QR$; ABC and ABC' are plane $\angle$ s of dihedral $\angle$ s formed by planes. (§ 379)

2. Then, diedral $\angle PRQN$ and $PRQM$ are equal. (§ 386)
3. Then, $\angle ABC = \angle ABC'$, or $\angle ABC$ is a rt. $\angle$. (§ 385)
4. Then, $AB \perp MN$, being $\perp$ to BC and BQ . (§ 352)

388. *If two planes are perpendicular, a perpendicular to one of them at any point of their intersection lies in the other.*

Let planes PQ and MN (Fig. of Prop. XXIII) be $\perp$, and intersect in line QR , and line AB be drawn from any point B of $QR \perp MN$.

A line drawn in PQ from $B \perp QR$ is $\perp MN$. (§ 387)

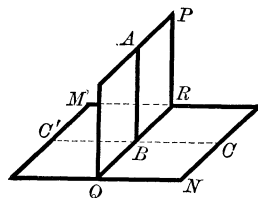
From B but one $\perp$ can be drawn to MN (§ 351); whence AB lies in PQ .

389. *If two planes are perpendicular, a perpendicular to one of them from any point of the other lies in the other.*

(The proof is left to the pupil; use figure of Prop. XXIII.)

PROP. XXIV. THEOREM

390. *If a straight line is perpendicular to a plane, every plane drawn through the line is perpendicular to the plane.*



Given line $AB \perp$ plane MN , and PQ any plane drawn through AB , intersecting MN in QR .

To Prove $PQ \perp MN$.

Proof. 1. Draw line $C'BC$ in $MN \perp QR$.

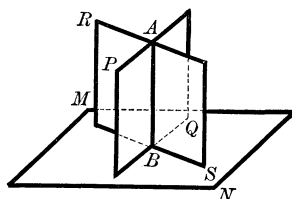
2. Since $AB \perp BQ$ (§ 350), ABC and ABC' are plane $\angle$ s of diedral $\angle PQRN$ and $PRQM$, respectively.

3. Then, $\angle ABC$ and ABC' are equal, being rt. $\angle$ s. (§ 350)

4. Then, diedral $\angle PRQN$, $PRQM$ are equal (§ 383), or $PQ \perp MN$.

PROP. XXV. THEOREM

391. *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*



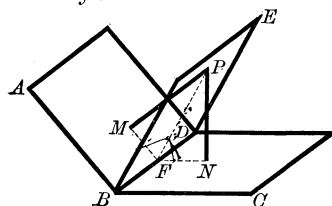
Given planes $PQ, RS \perp MN$, intersecting in line AB .

To Prove $AB \perp MN$.

(By § 388, a $\perp$ to MN at B lies in both PQ and RS .)

PROP. XXVI. THEOREM

392. *Every point in the bisecting plane of a dihedral angle is equally distant from its faces.*



Given P any point in bisecting plane BE of dihedral $\angle ABDC$, and lines PM and $PN \perp$ to AD and CD , respectively.

To Prove $PM = PN$.

Proof. 1. Let the plane of PM and PN intersect planes AD , BE , and CD in lines FM , FP , and FN , respectively.

2. We have plane $PFMN \perp AD$ and CD , or $\perp BD$.

(§§ 390, 391)

3. Then, $\angle PFM = \angle PFN$; being plane $\angle$ of dihedral $\angle ABDE$ and $CBDE$, respectively. (§§ 381, 385)

4. Also, $\angle PMF$ and PNF are rt. $\angle$. (§ 350)

5. By § 60, rt. $\triangle PMF, PNF$ are equal ($PF = PF$), and $PM = PN$.

PROP. XXVII. THEOREM

393. (Converse of Prop. XXVI.) *Any point which is within a dihedral angle, and equally distant from its faces, lies in the bisecting plane of the dihedral angle.*

Given point P within dihedral $\angle ABDC$, equally distant from AD and CD , and plane BE determined by BD and P . (Fig. of Prop. XXVI.)

To Prove that BE bisects dihedral $\angle ABDC$.

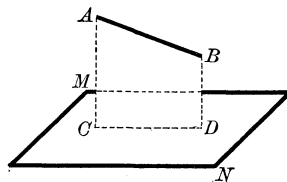
(Prove $\triangle PFM = \triangle PFN$, and $\angle PFM = \angle PFN$; then use § 383.)

394. It follows from §§ 392 and 393 that

The locus of points in space equally distant from the faces of a dihedral angle is the plane bisecting the dihedral angle.

PROP. XXVIII. THEOREM

395. *Through a given straight line without a plane, a plane can be drawn perpendicular to the given plane, and but one, unless the line is perpendicular to the plane.*



Given line AB without plane MN , not $\perp MN$.

To Prove that a plane can be drawn through $AB \perp MN$, and but one.

Proof. 1. Draw $AC \perp MN$; then $ACDB \perp MN$ by § 390.

2. If more than one plane could be drawn, their common intersection, AB , would be $\perp MN$ (§ 391), which is impossible.

[If AB is $\perp MN$, an indefinitely great number of planes can be drawn through $AB \perp MN$, by § 390.]

396. Defs. The *projection of a point on a plane* is the foot of the perpendicular drawn from the point to the plane.

The *projection of a line on a plane* is the line which contains the projections of all its points.

397. *The projection of a straight line on a plane is a straight line.*

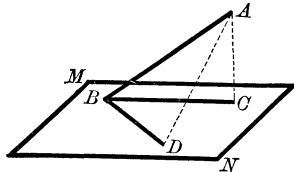
Let line CD (Fig. of Prop. XXVIII) be the projection of str. line AB on plane MN ; draw a plane through $AB \perp MN$.

By § 389, the $\perp$ to MN from all points of AB will lie in this plane; whence, CD is a str. line. (§ 348)

Ex. 16. Two parallel planes are cut by a third plane. Prove that the corresponding dihedral angles are equal.

PROP. XXIX. THEOREM

398. *The angle between a straight line and its projection on a plane is the least angle which it makes with any line drawn in the plane through its foot.*



Given line BC the projection of line AB on plane MN , and BD any other line drawn through B in MN .

To Prove $\angle ABC < \angle ABD$.

Proof. 1. Take $BD = BC$; draw lines AC , AD .

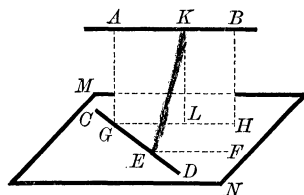
2. In $\triangle ABC$, ABD , $AB = AB$, and $BC = BD$.

3. We have $AC < AD$ (§ 360); the result follows by § 101

We call $\angle ABC$ the angle between line AB and plane MN .

PROP. XXX. THEOREM

399. *Two straight lines, not in the same plane, have one common perpendicular, and but one; and this line is the shortest line that can be drawn between them.*



Given lines AB and CD , not in the same plane.

To Prove that one common $\perp$ to AB and CD can be drawn, and but one; and that this line is the shortest line that can be drawn between AB and CD .

Proof. 1. Draw plane MN through $CD \parallel AB$. (§ 366)

2. Draw plane AH through $AB \perp MN$ (§ 395); let their intersection meet CD at G , and draw AG in plane $AH \perp GH$.

3. We have $AG \perp MN$ (§ 387); then, $AG \perp CD$. (?)

4. Now $AB \parallel GH$ (§ 362); then, $AG \perp AB$. (?)

5. If possible, let EK be another $\perp$ to AB and CD ; draw $EF \parallel AB$, and KL in plane $AH \perp GH$; then EF lies in MN . (§ 363)

6. Then $EK \perp ED$ and EF (§ 72); and $EK \perp MN$. (?)

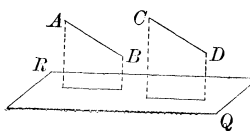
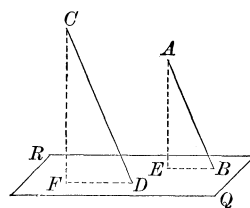
7. Since $KL \perp MN$ (§ 387), we have two $\perp$ s from K to MN , which is impossible; then only one common $\perp$ can be drawn.

8. Again, $EK > KL$, or $> AG$ (§§ 360, 96); this proves last statement of theorem.

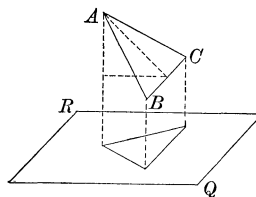
Ex. 17. Two parallel lines make equal angles with a plane, and their projections on the plane have the same ratio as the lines.

Ex. 18. Two parallel lines, AB and CD , are projected on a plane QR . The ratio of the projections is $\frac{3}{4}$. The projection of CD is 9. AB is 14. Find CD .

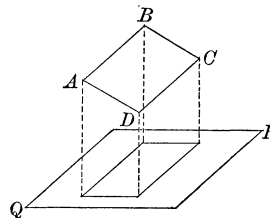
Ex. 19. A line AB makes an angle of 30° with the plane QR . The distances from A and B to QR are 16 and 8, respectively. CD is parallel to AB , and its projection on QR is 14. Find AB and CD .



Ex. 20. An equilateral $\triangle ABC$, whose area is $25\sqrt{3}$, has its side BC parallel to a plane QR . The plane of $\triangle ABC$ makes an angle of 45° with plane QR . Find the area of the projection of $\triangle ABC$ on plane QR .



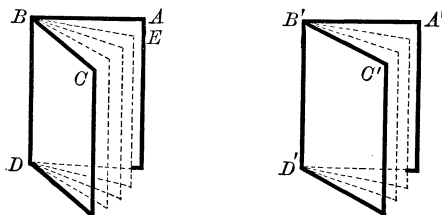
Ex. 21. The length of a rectangle $ABCD$ is 10, and its width 8. Side 10 is parallel to plane QR . Side 8 makes an angle of 30° with QR . Find area of the projection of $\square ABCD$ on plane QR , correct to three decimal places.



PROP. XXXI. THEOREM

400. Two diedral angles are to each other as their plane angles.

Case I. When the plane angles are commensurable.



Given ABC and $A'B'C'$, plane $\angle$ of diedral $\angle$ $ABDC$ and $A'B'D'C'$, respectively, and commensurable.

To Prove
$$\frac{ABDC}{A'B'D'C'} = \frac{\angle ABC}{\angle A'B'C'}. \quad (1)$$

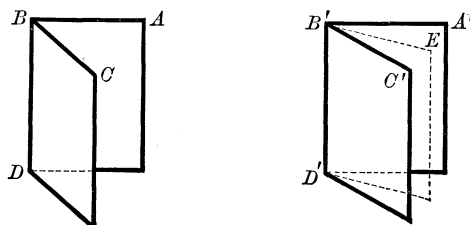
Proof. 1. Let $\angle ABE$ be contained 4 times in $\angle ABC$, 3 times in $\angle A'B'C'$; find ratio $\angle ABC$ to $\angle A'B'C'$.

2. Pass planes through edges of diedrals, and division lines of plane $\angle$; subdivisions of diedrals are all equal. (§ 383)

3. Find ratio diedral $ABDC$ to diedral $A'B'D'C'$.

4. From results (1) and (3), we have equation (1).

Case II. When the plane angles are incommensurable.



Given ABC and $A'B'C'$ plane $\angle$ s of dihedral $\angle$ s $ABDC$ and $A'B'D'C'$, respectively, and incommensurable.

To Prove $\frac{ABDC}{A'B'D'C'} = \frac{\angle ABC}{\angle A'B'C'}$. (2)

- Proof.** 1. Divide $\angle ABC$ into any number of equal parts.
 2. Let one of the parts be contained exactly in $\angle A'B'E$, with a remainder $\angle C'B'E <$ one of the parts.
 3. Pass plane through $B'D'$ and $B'E$.
 4. By Case I, $\frac{ABDC}{A'B'D'E} = \frac{\angle ABC}{\angle A'B'E}$.
 5. If number of subdivisions of $\angle ABC$ be indefinitely increased, find limits $\frac{ABDC}{A'B'D'E}$ and $\frac{\angle ABC}{\angle A'B'E}$.
 6. We have equation (2) by equating the limits. (§ 187)

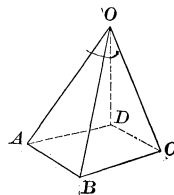
By § 400, the plane angle may be taken as the *measure* of the dihedral angle.

POLYEDRAL ANGLES

DEFINITIONS

401. A *polyedral angle* is a figure consisting of three or more triangles, called *faces*, having for their bases the sides of a polygon, and for their common vertex a point without its plane; as $O-ABCD$.

We call O the *vertex* of the polyedral, and polygon $ABCD$ the *base*; we call $\angle AOB$, BOC , etc., the *face angles*, and their sides AO , OB , etc., the *edges*.



The polyedral angle is not regarded as limited by the base; thus, the face AOB means the plane between OA and OB extended indefinitely.

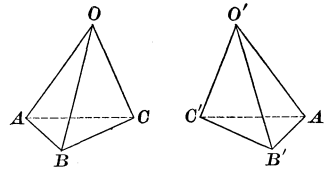
A *triedral angle* is a polyedral angle of three faces.

Two polyedral angles are called *vertical* when the edges of one are the prolongations of the edges of the other.

402. A polyedral angle is called *convex* when its base is a convex polygon (§ 118).

403. Two polyedral angles are *equal* when they can be applied to each other so that their faces shall coincide.

404. Two polyedral angles are said to be *symmetrical* when the face and diedral angles of one are equal respectively to the homologous face and diedral angles of the other, if the equal parts occur *in the reverse order*.



Thus, if face $\angle AOB$, BOC , and COA are equal respectively to face $\angle A'O'B'$, $B'O'C'$, and $C'O'A'$, and diedral $\angle OA$, OB , and OC to diedral $\angle O'A'$, $O'B'$, and $O'C'$, triedral $\angle O-ABC$ and $O'-A'B'C'$ are symmetrical.

PROP. XXXII. THEOREM

405. *Two vertical polyedral angles are symmetrical.*

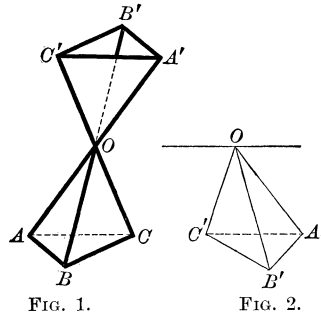


FIG. 1.

FIG. 2.

Given $O-ABC$ and $O-A'B'C'$ (Fig. 1) vertical triedral $\angle$.

To Prove $O-ABC$ and $O-A'B'C'$ symmetrical.

Proof. 1. The face $\angle$ s of the vertical triedrals are equal. (§ 44)

2. Diedrals OA and OA' are vertical (§ 378); for faces AOB and $A'OB'$ are in one plane, as also are faces AOC and $A'OC'$.

3. Then diedrals OA , OA' are equal. (§ 384)

4. Similarly, diedrals OB and OB' , as also OC and OC' are equal.

5. Since equal parts are *in reverse order*, triedrals are symmetrical. (§ 404)

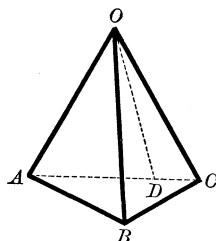
That the equal parts of the triedrals occur in reverse order may be seen by moving $O-A'B'C'$ $\parallel$ to itself to the right, and then revolving it, as shown in Fig. 2, about an axis passing through O , until face $OA'C'$ comes into the same plane as before; edge OB' being on this side of, instead of beyond, plane $OA'C'$.

In like manner, the theorem may be proved for two polyedral $\angle$ s, having any number of faces.

PROP. XXXIII. THEOREM

406. *The sum of any two face angles of a triedral angle is greater than the third.*

The theorem requires proof only in the case where the third face angle is greater than either of the others.



Given in triedral $\angle O-ABC$,

face $\angle AOC >$ face $\angle AOB$ or face $\angle BOC$.

To Prove $\angle AOB + \angle BOC > \angle AOC$. (1)

Proof. 1. In the face AOC take line $OD = OB$, making $\angle AOD = \angle AOB$; pass plane through B, D cutting faces in AB, BC, CA .

2. We have $\triangle AOB = \triangle AOD$ (§ 49); whence $AB = AD$.

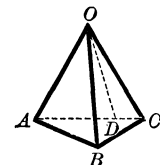
3. By Ax. 7, $AB + BC > AD + DC$.

4. From results (2) and (3), $BC > DC$.

5. From results (2), (4), and $OB = OD$,

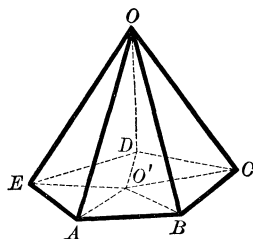
$$\angle BOC > \angle COD. \quad (\S 101)$$

6. Adding $\angle AOB$ to first member of this, and equal $\angle AOD$ to second member, and putting $\angle AOC$ for $\angle AOD + \angle COD$, gives inequality (1).



PROP. XXXIV. THEOREM

407. *The sum of the face angles of any convex polyedral angle is less than four right angles.*



Given $O-ABCDE$ a convex polyedral $\angle$.

To Prove $\angle AOB + \angle BOC + \text{etc.} < 4 \text{ rt. } \angle$.

Proof. 1. Draw lines from any point O' , in base of polyedral, to vertices A, B , etc.

2. We have $\angle OAE + \angle OAB > \angle O'AE + \angle O'AB$.

(§ 406)

3. Similarly $\angle OBA + \angle OBC > \angle O'BA + \angle O'BC$, etc.

4. Adding inequalities, sum of base $\angle$ s of $\triangle OAB, OBC$, etc., is $>$ sum of base $\angle$ s of $\triangle O'AB, O'BC$, etc.

5. Since sum of *all* $\angle$ s of $\triangle OAB, OBC$, etc., equals sum of *all* $\angle$ s of $\triangle O'AB, O'BC$, etc. (§ 84), sum of $\angle$ s at O is $<$ sum of $\angle$ s at O' , which proves theorem.

PROP. XXXV. THEOREM

408. *If two triedral angles have the face angles of one equal respectively to the face angles of the other, their homologous diedral angles are equal.*

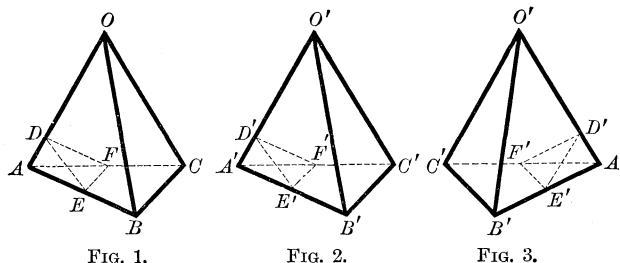


FIG. 1.

FIG. 2.

FIG. 3.

Given, in trihedral $\angle O-ABC$ and $O'-A'B'C'$,

$$\angle AOB = \angle A'O'B', \angle BOC = \angle B'O'C',$$

$$\text{and } \angle COA = \angle C'O'A'.$$

To Prove diedral $\angle OA =$ diedral $\angle O'A'$.

Proof. 1. Lay off $OA, OB, OC, O'A', O'B', O'C'$ all equal; draw lines $AB, BC, CA, A'B', B'C', C'A'$.

2. We have $\triangle OAB = \triangle O'A'B'$ (§ 46); then $AB = A'B'$. (?)

3. In like manner, $BC = B'C', CA = C'A'$.

4. Then $\triangle ABC = \triangle A'B'C'$ (§ 52); and $\angle EAF = \angle E'A'F'$. (?)

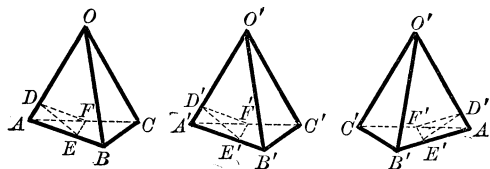
5. Take $AD = A'D'$; and draw DE in face $OAB \perp OA$, meeting AB at E (it will meet, since $\triangle OAB$ is isosceles, and $\angle OAB$ acute).

6. Draw DF in face $OAC \perp OA$, meeting AC at F ; and $D'E'$ and $D'F'$ in faces $O'A'B', O'A'C', \perp O'A'$, meeting $A'B'$ at E' , and $A'C'$ at F' ; and draw lines $EF, E'F'$.

7. In rt. $\triangle ADE, A'D'E', AD = A'D'$; and from equal $\triangle OAB, O'A'B', \angle DAE = \angle D'A'E'$. (?)

8. Then $\triangle ADE = \triangle A'D'E'$ (§ 89); whence, $AE = A'E'$, and $DE = D'E'$.

9. In like manner, $AF = A'F', DF = D'F'$.



10. From results (4), (8) and (9), $\triangle AEF = \triangle A'E'F'$, and $EF = E'F'$.

11. From results (8), (9) and (10), $\triangle DEF = \triangle D'E'F'$. (§ 52)

12. Then, $\angle EDF = \angle E'D'F'$ (?); these are plane $\angle$ s of dihedrals OA and $O'A'$ (§ 379), whence the dihedrals are equal by § 383.

The above proof holds for Fig. 3 as well as for Fig. 2; in Figs. 1 and 2, the equal parts occur in the *same* order, and in Figs. 1 and 3 in the *reverse* order.

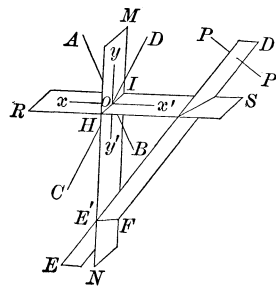
409. It follows from § 408 that *if two trihedral angles have the face angles of one equal respectively to the face angles of the other,*

1. *They are equal if the equal parts occur in the same order.*

For if trihedral $\angle O'-A'B'C'$ (Fig. 2) be applied to $O-ABC$ so that dihedral $\angle O'A'$ and OA coincide, point O' falling at O , then since $\angle A'O'C' = \angle AOC$, and $\angle A'O'B' = \angle AOB$, $O'B'$ will coincide with OB , and $O'C'$ with OC .

2. *They are symmetrical if the equal parts occur in the reverse order.*

Ex. 22. What is the locus of points equidistant from two intersecting lines, and also equidistant from two given points?



BOOK VII

POLYEDRONS

DEFINITIONS

410. A *polyedron* is a solid bounded by polygons.

We call the bounding polygons the *faces* of the polyedron, their sides the *edges*, and their vertices the *vertices*; a *diagonal* is a straight line joining two vertices not in the same face.

The least number of polygons which can bound a polyedron is four.

A polyedron of four faces is called a *tetraedron*; of six faces, a *hexaedron*; of eight faces, an *octaedron*; of twelve faces, a *dodecaedron*; of twenty faces, an *icosaedron*.

411. A polyedron is called *convex* when the section made by any plane is a convex polygon (§ 118).

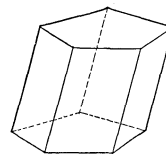
All polyedrons considered hereafter will be understood to be convex.

412. Any two solids are called *equivalent* when they have the same volume.

PRISMS AND PARALLELOPIPEDS

413. A *prism* is a polyedron, two of whose faces are equal polygons lying in parallel planes, having their homologous sides parallel, the other faces being parallelograms (§ 107).

We call the equal and parallel faces the *bases* of the prism, and the others the *lateral faces*; we call the edges which are not sides of the bases the *lateral edges*, and the sum of the areas of the lateral faces the *lateral area*.



The *altitude* is the perpendicular distance between the planes of the bases.

414. The following is given for reference:

The bases of a prism are equal.

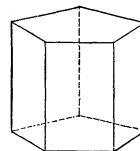
415. It follows from § 413 that *the lateral edges of a prism are equal and parallel.* (§ 104)

416. A prism is called *triangular*, *quadrangular*, etc., according as its base is a triangle, quadrilateral, etc.

417. A *right prism* is a prism whose lateral edges are perpendicular to its bases.

The lateral faces are rectangles (§ 350).

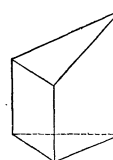
An *oblique prism* is a prism whose lateral edges are not perpendicular to its bases.



418. A *regular prism* is a right prism whose base is a regular polygon.

419. A *truncated prism* is a portion of a prism included between the base, and a plane, not parallel to the base, cutting all the lateral edges.

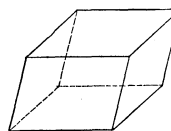
The base of the prism and the section made by the plane are called the *bases* of the truncated prism.



420. A *right section* of a prism is a section made by a plane cutting all the lateral edges, and perpendicular to them.

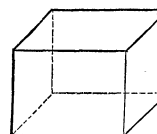
421. A *parallelepiped* is a prism whose bases are parallelograms; that is, all the faces are parallelograms.

A *right parallelepiped* is a parallelepiped whose lateral edges are perpendicular to its bases.



A *rectangular parallelepiped* is a right parallelepiped whose bases are rectangles; that is, all the faces are rectangles.

A *cube* is a rectangular parallelepiped whose six faces are all squares.



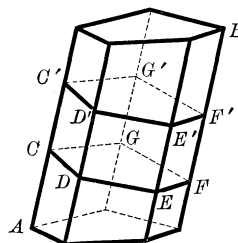
PROP. I. THEOREM

422. *The sections of a prism made by two parallel planes which cut all the lateral edges, are equal polygons.*

Given $\parallel$ planes CF and $C'F'$ cutting all the lateral edges of prism AB .

To Prove

section $CDEFG = C'D'E'F'G'$.

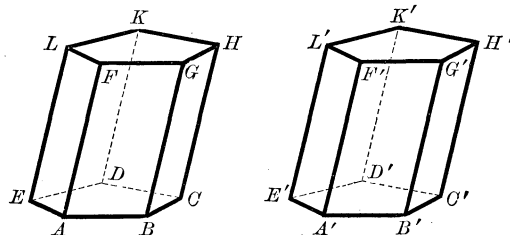


Proof. 1. The homologous sides $CD, C'D'$, etc., are $\parallel$. (§ 364)

2. Then, the homologous sides and $\angle$ s are equal (§§ 105, 1; 376); then use § 123.

PROP. II. THEOREM

423. *Two prisms are equal when the faces including a triedral angle of one are equal respectively to the faces including a triedral angle of the other, and similarly placed.*



Given, in prisms AH and $A'H'$, faces $ABCDE, AG$, and AL equal respectively to faces $A'B'C'D'E', A'G'$, and $A'L'$; the equal parts being similarly placed.

To Prove prism $AH =$ prism $A'H'$.

Proof. 1. Since $\angle EAB = \angle E'A'B'$, $\angle EAF = \angle E'A'F'$, $\angle FAB = \angle F'A'B'$ (§ 48) triedral $\angle A-BEF, A'-B'E'F'$ are equal. (§ 409, 1)

2. Apply prisms so that $A'B'C'D'E', A'G'$, and $A'L'$ shall coincide with $ABCDE, AG$, and AL , respectively.

3. Then edges $C'H', D'K'$ fall on CH, DK , and vertices H', K' on H, K . (§§ 69, 415)

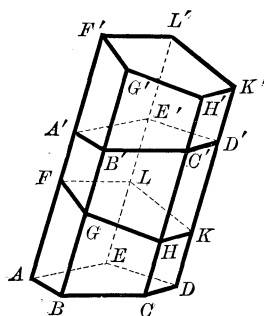
4. Then faces $B'H', C'K', D'L', H'L'$ coincide with BH, CK, DL, HL (§§ 346, 347); and prisms are equal.

424. *Two right prisms are equal when they have equal bases and equal altitudes ; for by inverting one of the prisms if necessary, the equal faces will be similarly placed.*

425. Note. The proof in §423 holds for two *truncated prisms*.

PROP. III. THEOREM

426. *An oblique prism is equivalent to a right prism, having for its base a right section of the oblique prism, and for its altitude a lateral edge of the oblique prism.*



Given FK' a right prism, having for its base FK a right section of oblique prism AD' , and its altitude FF' equal to AA' , a lateral edge of AD' .

To Prove

$$AD' \approx FK'.$$

Proof. 1. In truncated prisms AK , $A'K'$ faces AD , $A'D'$ are equal (?); then, $AB = A'B'$.

2. By § 422, FK , $F'K'$ are equal; then, $FG = F'G'$.

3. Since $FF' = AA'$, $AF = A'F'$; similarly, $BG = B'G'$.

4. Also homologous $\angle$ s of AG , $A'G'$ are equal. (§§ 413, 76)

5. Then $AG = A'G'$ (§ 123); similarly, $BH = B'H'$.

6. Then $AK = A'K'$. (§§ 423, 425)

7. Taking AK from solid AK' , we get FK' , and taking $A'K'$, we get AD' ; then $AD' \approx FK'$.

PROP. IV. THEOREM

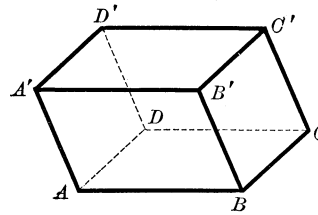
427. *The opposite lateral faces of a parallelepiped are equal and parallel.*

Given AC and $A'C'$ the bases of parallelepiped AC' .

To Prove AB' and DC' , also AD' and BC' , equal and $\parallel$.

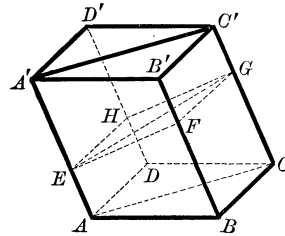
Proof. 1. Since AB is equal and $\parallel$ to DC , and AA' to DD' , $\angle A'AB = \angle D'DC$, and $AB' \parallel DC'$. (§ 376)

2. Then $AB' = DC'$ (§ 110); similarly for AD' , BC' .



PROP. V. THEOREM

428. *The plane passed through two diagonally opposite edges of a parallelepiped divides it into two equivalent triangular prisms.*



Given plane AC' passing through edges AA' and CC' of parallelepiped $A'C'$.

To Prove prism $ABC-A'$ $\cong$ prism $ACD-A'$. (1)

Proof. 1. Let rt. section $EFGH$ intersect plane AC' in EG .

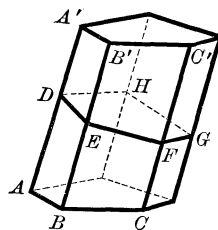
2. Faces AB' , DC' are $\parallel$, and $EF \parallel GH$. (§§ 427, 364)

3. Also, $EH \parallel FG$; $EFGH$ is a $\square$, and $\triangle EFG = \triangle EGH$. (?)

4. By § 426, $ABC-A'$ is $\cong$ a rt. prism of base EFG and altitude AA' , and $ACD-A'$ to a rt. prism of base EGH and altitude AA' ; also, the rt. prisms are equal, having equal bases and altitudes (§ 424), which proves (1).

PROP. VI. THEOREM

429. *The lateral area of a prism is equal to the perimeter of a right section multiplied by a lateral edge.*



Given $DEFGH$ a rt. section of prism AC' .

To Prove lat. area $AC' = (DE + EF + \text{etc.}) \times AA'$.

Proof. 1. We have $DE \perp AA'$, $EF \perp BB'$, etc. (§ 350)

2. By §§ 282, 415,

$$\begin{aligned} \text{area } AB' + \text{area } BC' + \text{etc.} &= DE \times AA' + EF \times AA', \text{ etc.} \\ &= (DE + EF + \text{etc.}) \times AA'. \end{aligned}$$

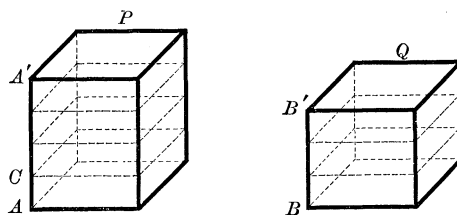
430. It follows from § 429 that *the lateral area of a right prism is equal to the perimeter of the base multiplied by the altitude.*

PROP. VII. THEOREM

431. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*

The phrase “rectangular parallelopiped” in the above statement signifies the *volume* of the rectangular parallelopiped.

Case I. *When the altitudes are commensurable.*



Given P and Q rect. parallelopipeds, with equal bases, and commensurable altitudes, AA' and BB' .

To Prove
$$\frac{P}{Q} = \frac{AA'}{BB'}. \quad (1)$$

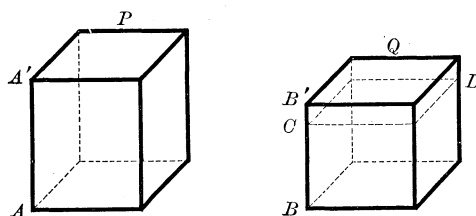
Proof. 1. Let AC be contained 4 times in AA' , 3 times in BB' ; find ratio AA' to BB' .

2. Pass planes through points of division of AA' , BB' , $\perp$ to AA' , BB' ; subdivisions of P and Q are all equal. (§ 424)

3. Find ratio P to Q .

4. From results (1) and (3), we have equation (1).

Case II. When the altitudes are incommensurable.



Given P and Q rect. parallelopipeds, with equal bases, and incommensurable altitudes, AA' and BB' .

To Prove
$$\frac{P}{Q} = \frac{AA'}{BB'}. \quad (2)$$

Proof. 1. Divide AA' into any number of equal parts; let one of the parts be contained a certain number of times in BB' , with a remainder $CB' <$ one of the parts.

2. Pass plane $CD \perp BB'$, forming rect. parallelopiped BD .

3. By Case I,
$$\frac{P}{BD} = \frac{AA'}{BC}.$$

4. If number of subdivisions of AA' be indefinitely increased, find limits of $\frac{P}{BD}$ and $\frac{AA'}{BC}$.

5. We have equation (2) by equating limits. (§ 187)

Ex. 1. The base of a right prism is a square whose side is 6; if the lateral edge of this prism is 10, what is the lateral area? If the diagonal of the base is 6, what is the lateral area?

432. Note. The *dimensions* of a rectangular parallelepiped are the three edges meeting at any vertex.

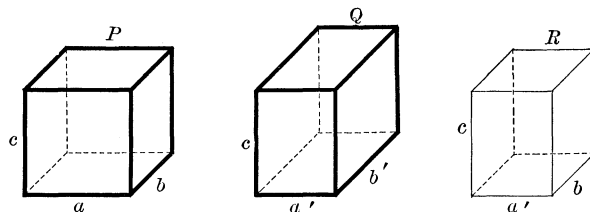
Then, the theorem of § 431 may be expressed:

If two rectangular parallelepipeds have two dimensions of one equal respectively to two dimensions of the other, they are to each other as their third dimensions.

Ex. 2. If the base of a right prism is an equilateral triangle whose side is 4, and the lateral edge of the prism is 10, what is the lateral area and total area? If the altitude of the base is 4, what are the above areas?

PROP. VIII. THEOREM

433. *Two rectangular parallelepipeds having equal altitudes are to each other as their bases.*



Given P and Q rectangular parallelepipeds, with altitude c , and dimensions of the bases a, b , and a', b' , respectively.

To Prove
$$\frac{P}{Q} = \frac{a \times b}{a' \times b'} \quad (\S\ 280). \quad (1)$$

Proof. 1. If R is a rectangular parallelepiped with altitude c , and base dimensions a', b , P and R have each the dimensions b, c , and are as their third dimensions (§ 432).

2. Then
$$\frac{P}{R} = \frac{a}{a'}. \quad (2)$$

3. Since R and Q have dimensions a', c ,

$$\frac{R}{Q} = \frac{b}{b'}. \quad (3)$$

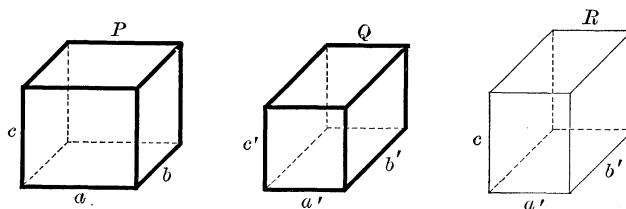
4. Multiplying (2) and (3), and cancelling R , gives (1).

434. Note. The theorem of § 433 may be expressed :

Two rectangular parallelepipeds having a dimension of one equal to a dimension of the other, are to each other as the products of their other two dimensions.

PROP. IX. THEOREM

435. *Any two rectangular parallelepipeds are to each other as the products of their three dimensions.*



Given P and Q rect. parallelepipeds with the dimensions a, b, c , and a', b', c' , respectively.

To Prove
$$\frac{P}{Q} = \frac{a \times b \times c}{a' \times b' \times c'}$$

(If R is a rect. parallelepiped with dimensions a', b', c , find values of $\frac{P}{R}$ and $\frac{R}{Q}$ by §§ 434, 432.)

Ex. 3. Find the lateral area of a right prism whose base is a regular hexagon, if one of the lateral faces has the base 10, and altitude 8.

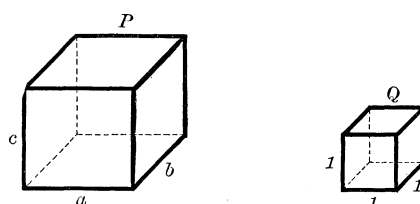
Ex. 4. The edges of two rectangular parallelepipeds are 9, 12, 15, and 15, 12, 18, respectively. Find the ratio of their volumes.

436. Def. The *volume* of a solid is its ratio to another solid, called the *unit of volume*, adopted arbitrarily as the unit of measure (§ 180).

The usual unit of volume is a cube (§ 421) whose edge is some linear unit ; for example, a *cubic inch* or a *cubic foot*.

PROP. X. THEOREM

437. *If the unit of volume is the cube whose edge is the linear unit, the volume of a rectangular parallelopiped is equal to the product of its three dimensions.*



Given a , b , and c the dimensions of rect. parallelopiped P , and Q the unit of volume; that is, a cube whose edge is 1.

To Prove $\text{vol. } P = a \times b \times c.$

Proof. 1. We have $\frac{P}{Q} = \frac{a \times b \times c}{1 \times 1 \times 1}.$ (§ 435)

2. Since Q is unit of volume, $\frac{P}{Q}$ is volume $P.$ (§ 436)

438. Note. In all succeeding theorems relating to volumes, it is understood that the *unit of volume* is the cube whose edge is the linear unit, and the *unit of surface* the square whose side is the linear unit.

The following are direct consequences of § 437:

439. *The volume of a cube equals the cube of its edge.*

440. *The volume of a rectangular parallelopiped equals the product of its base and altitude.*

Ex. 5. The volume of a cube is 64 cubic inches. Using proportion, find the edge of a cube having eight times the volume.

Ex. 6. A cubical tank holds 30 gallons of water; find an edge of the tank in feet.

Ex. 7. Find the ratio of a diagonal of a cube to an edge. Is this ratio commensurable?

Ex. 8. The volume of a cube is $16\sqrt{3}$. Find the diagonal.

Ex. 9. The diagonal of a cube is $D\sqrt{3}$. Find the volume.

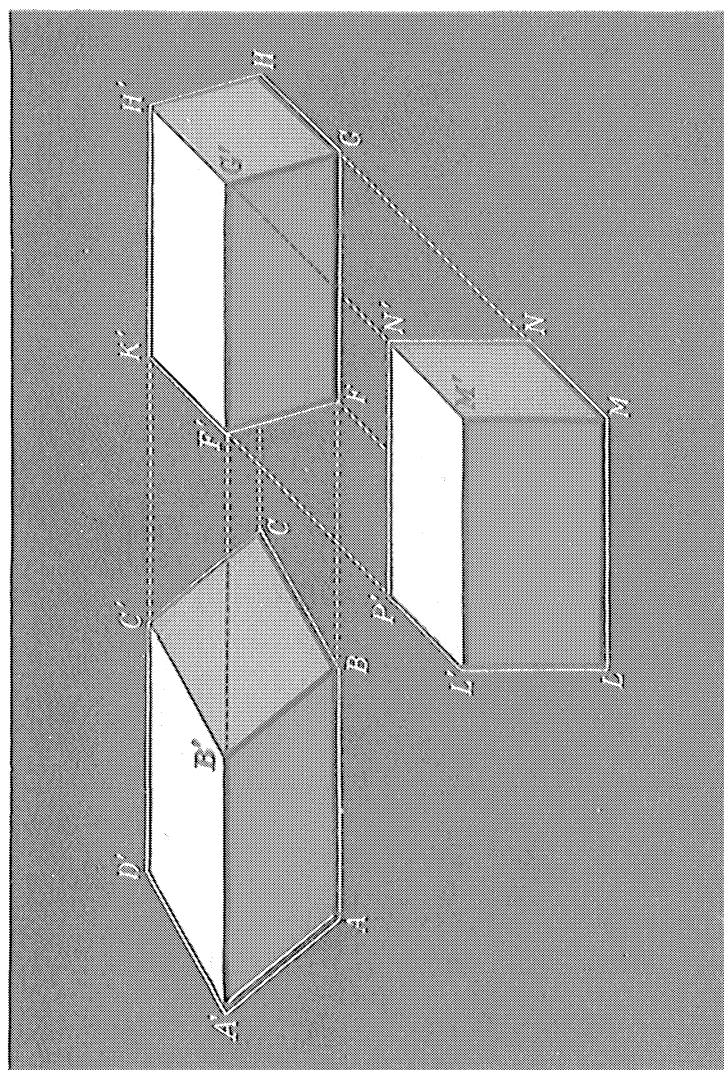
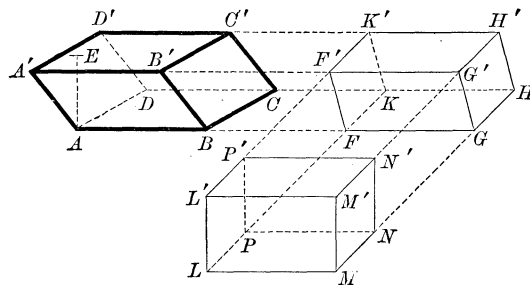


PLATE VIII

PROP. XI. THEOREM

441. *The volume of any parallelopiped is equal to the product of its base and altitude.*



Given AE the altitude of parallelopiped AC' .

To Prove $\text{vol. } AC' = \text{area } ABCD \times AE$.

Proof. 1. On AB prolonged take $FG = AB$; draw planes FK' , $GH' \perp FG$, meeting $A'B'$, $D'C'$, DC prolonged, forming rt. parallelopiped $FH' \approx AC'$. (§ 426)

2. On HG prolonged, take $NM = HG$; draw planes NP' , $ML' \perp NM$, meeting $H'G'$, $K'F'$, KF prolonged, forming rt. parallelopiped $LN' \approx FH'$ (§ 426), and hence $\approx AC'$.

3. Since FG is $\perp GH'$, planes LH , MH' are $\perp$. (§ 390)

4. Since MM' is $\perp MN$, it is $\perp$ plane LH . (§ 387)

5. Then LMM' is a rt. $\angle$, and LM' a rectangle (§§ 350, 77); whence LN' is a rect. parallelopiped.

6. Then $\text{vol. } LN'$ (or AC') = $\text{area } LMNP \times MM'$. (§ 440)

7. Now rect. $LN = \text{rect. } FH$, having equal bases MN , GH , and same altitude (§ 111); also, rect. $FH \approx \square AC$, having equal bases FG , AB , and same altitude (§ 283); then, $LN \approx AC$.

8. Substitute in result of (6) area $ABCD$ for area $LMNP$, and put $MM' = AE$ (§ 374).

Ex. 10. If the edge of a cube equals a , find the diagonal of a face, also the diagonal of the cube.

Ex. 11. The diagonal of a cube is 12. Find its edge.

Ex. 12. The diagonal of a cube is d . Find its volume.

Ex. 13. The area of the surface of a cube is 1 square foot. Find its edge and the edge of a cube the area of whose surface is 2 square feet.

Ex. 14. A bin in the form of a rectangular parallelopiped must contain $328\frac{1}{3}$ bushels and is 6 feet in height. Find number of square feet of material necessary for construction. Will a change in the form of the base make any change in the amount of material used?

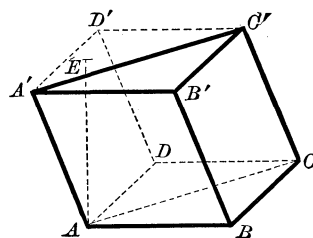
Ex. 15. The diagonal of a cubical box is $18\sqrt{3}$ inches. How many bushels of grain will it contain?

Ex. 16. Pine lumber, green, weighs about 830 Kg. per cubic meter, kiln dried, 460 Kg. per cubic meter. Find the weight of a pile of 16-foot kiln-dried lumber 8 ft. wide and 12 ft. high, allowing one-tenth of the volume for air spaces.

Ex. 17. If the diagonals of a rectangular parallelopiped meet in a point, any line through this point terminating in the surface of the parallelopiped is bisected by this point. Is this true of any parallelopiped?

PROP. XII. THEOREM

442. *The volume of a triangular prism is equal to the product of its base and altitude.*



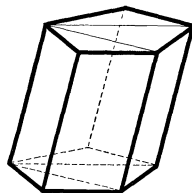
Given AE the altitude of triangular prism $ABC-C'$.

To Prove $\text{vol. } ABC-C' = \text{area } ABC \times AE.$

Proof. If edges of parallelopiped $ABCD-D'$ be $\parallel AB, BC, BB'$, respectively, by § 428, $ABC-C' \approx \frac{1}{2} ABCD-D'$; by § 441, $\text{vol. } ABC-C' = \frac{1}{2} \text{area } ABCD \times AE = \text{area } ABC \times AE.$ (?)

PROP. XIII. THEOREM

443. *The volume of any prism is equal to the product of its base and altitude.*



Given any prism.

To Prove its volume equal to the product of its base and altitude.

Proof. 1. Divide the prism into triangular prisms by planes through a lateral edge and the diagonals of the base.

2. Since volume of each triangular prism equals product of its base and altitude (§ 442), the sum of the volumes of the triangular prisms equals the sum of their bases times their common altitude, which proves the theorem.

444. It follows from § 443 that

1. *Prisms having equivalent bases and equal altitudes are equivalent.*
2. *Two prisms having equal altitudes are to each other as their bases.*
3. *Two prisms having equivalent bases are to each other as their altitudes.*
4. *Any two prisms are to each other as the products of their bases by their altitudes.*

Ex. 18. The volume of a right prism whose base is a regular hexagon is 600. One side of the base is 10; find the altitude.

Ex. 19. The base of a right prism is an isosceles right triangle whose hypotenuse is c . The volume of the prism is v ; find the lateral area.

Ex. 20. The sides of a right section of an oblique triangular prism are 4, 5, and 7. The lateral edge is 14. Find the volume.

Ex. 21. The lateral edges of a prism whose base is a regular hexagon make an angle of 60° with the base. A perpendicular from the centre of the upper base meets the lower base at a point 10 feet from its centre. If a side of the lower base is 6, find the volume of the prism.

Ex. 22. A ditch is a right prism in form ; its length is 100 feet. Its ends are isosceles trapezoids whose bases are 10 feet and 3 feet, respectively. Its depth is 4 feet. Find its volume.

Ex. 23. The volume of a triangular prism is 288. The base is an equilateral triangle whose side is 8. Find the altitude of the prism.

PYRAMIDS

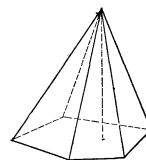
DEFINITIONS

445. A *pyramid* is a polyedron bounded by a polygon, called the *base*, and a series of triangles having a common vertex.

The common vertex of the triangular faces is called the *vertex* of the pyramid.

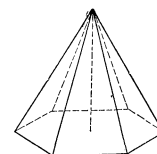
The triangular faces are called the *lateral faces*, the edges terminating at the vertex the *lateral edges*, and the sum of the areas of the lateral faces the *lateral area*.

The *altitude* is the perpendicular distance from the vertex to the plane of the base.



446. A pyramid is called *triangular*, *quadrangular*, etc., according as its base is a triangle, quadrilateral, etc.

A *regular pyramid* is a pyramid whose base is a regular polygon, and whose vertex lies in the perpendicular to the base, at its centre.

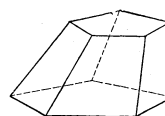


447. A *truncated pyramid* is a portion of a pyramid included between the base and a plane cutting all the lateral edges.

The base of the pyramid and the section made by the plane are called the *bases* of the truncated pyramid.

448. A *frustum of a pyramid* is a truncated pyramid whose bases are parallel.

The *altitude* is the perpendicular distance between the planes of the bases.



PROP. XIV. THEOREM

449. *In a regular pyramid, the lateral edges are equal, and the lateral faces are equal isosceles triangles.*

Given regular pyramid $O-ABCDEF$.

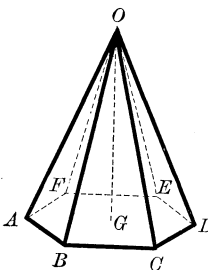
To Prove $OA = OB$, etc., and $\triangle OAB = \triangle OBC$, etc.

(Prove by §§ 356, I, and 52.)

450. Def. The *slant height* of a regular pyramid is the altitude of any lateral face.

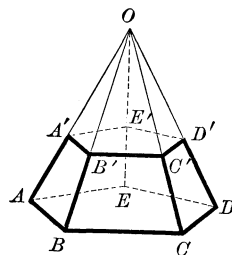
Or, it is the line drawn from the vertex of the pyramid to the middle point of any side of the base.

(§ 91)



PROP. XV. THEOREM

451. *The lateral faces of a frustum of a regular pyramid are equal trapezoids.*



Given AC' a frustum of regular pyramid $O-ABCDE$.

To Prove faces AB' , BC' , etc., equal trapezoids.

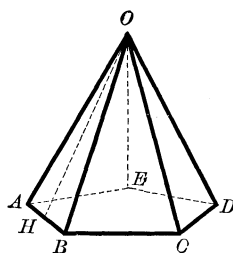
Proof. 1. Since $\triangle OAB = \triangle OBC$ (§ 449), we may apply $\triangle OAB$ to $\triangle OBC$ so that OB , OA , AB shall coincide with OB , OC , BC , respectively; also $A'B' \parallel AB$, $B'C' \parallel BC$. (?)

2. Then $A'B'$, $B'C'$ coincide (§ 69); this proves theorem.

452. It follows from § 451 that *the lateral edges of a frustum of a regular pyramid are equal.*

PROP. XVI. THEOREM

453. *The lateral area of a regular pyramid is equal to the perimeter of its base multiplied by one-half its slant height.*



Given slant height OH of regular pyramid $O-ABCDE$.

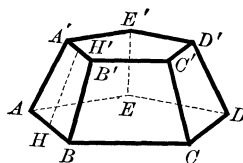
To Prove

$$\text{lat. area } O-ABCDE = (AB + BC + \text{etc.}) \times \frac{1}{2} OH.$$

(By § 450, OH is the altitude of each lateral face.)

PROP. XVII. THEOREM

454. *The lateral area of a frustum of a regular pyramid is equal to one-half the sum of the perimeters of its bases, multiplied by its slant height.*



Given slant height HH' of the frustum of a regular pyramid AD' .

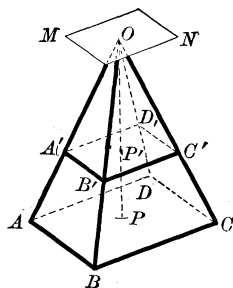
To Prove

$$\text{lat. area } AD' = \frac{1}{2} (AB + A'B' + BC + B'C' + \text{etc.}) \times HH',$$

(HH' is the altitude of each lateral face.)

PROP. XVIII. THEOREM

455. *If a pyramid be cut by a plane parallel to the base, the lateral edges and altitude are divided proportionally, and the section is similar to the base.*



Given plane $A'C' \parallel$ to base of pyramid $O-ABCD$, cutting faces OAB , OBC , OCD , and ODA in lines $A'B'$, $B'C'$, $C'D'$, and $D'A'$, respectively, and altitude OP at P' .

To Prove
$$\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC}, \text{ etc. } = \frac{OP'}{OP}, \quad (1)$$

and $A'B'C'D'$ similar to $ABCD$.

Proof. 1. To prove (1), pass plane MN through $O \parallel ABCD$, and use § 377.

2. Again, since $A'B' \parallel AB$, $B'C' \parallel BC$, etc. (?), homologous $\angle$ s of $A'B'C'D'$ and $ABCD$ are equal. (§ 376)

3. Also, $\triangle OA'B'$, OAB are similar (§ 238), and $\frac{OA'}{OA} = \frac{A'B'}{AB}$.

4. Also, $\frac{OB'}{OB} = \frac{B'C'}{BC}$, etc.; whence by (1), $\frac{A'B'}{AB} = \frac{B'C'}{BC}$, etc.

5. From results (2) and (4), $A'B'C'D'$, $ABCD$ are similar. (§ 233)

Ex. 24. All the faces of a pyramid are equal equilateral triangles. The altitude of the pyramid is $\sqrt{6}$. Find the edge.

456. Since $A'B'C'D'$ and $ABCD$ are similar,

$$\frac{\text{area } A'B'C'D'}{\text{area } ABCD} = \frac{\overline{A'B'}^2}{\overline{AB}^2}. \quad (\S 291)$$

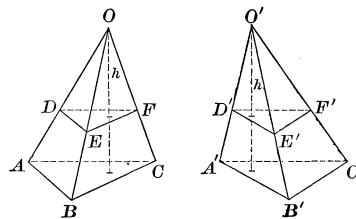
But from § 455,

$$\frac{A'B'}{AB} = \frac{OA'}{OA} = \frac{OP'}{OP}, \text{ whence } \frac{\text{area } A'B'C'D'}{\text{area } ABCD} = \frac{\overline{OP'}^2}{\overline{OP}^2}.$$

Hence, *the area of a section of a pyramid, parallel to the base, is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

457. *If two pyramids have equal altitudes and equivalent bases, sections parallel to the bases equally distant from the vertices are equivalent.*

Let pyramids $O-ABC$, $O'-A'B'C'$ have $\approx$ bases, and same altitude H ; let DEF , $D'E'F'$ be sections $\parallel$ to bases, at distance h from O , O' .



$$\text{By } \S 456, \frac{\text{area } DEF}{\text{area } ABC} = \frac{h^2}{H^2}, \text{ and } \frac{\text{area } D'E'F'}{\text{area } A'B'C'} = \frac{h^2}{H^2}.$$

$$\text{Then, } \frac{\text{area } DEF}{\text{area } ABC} = \frac{\text{area } D'E'F'}{\text{area } A'B'C'}. \quad (?)$$

Since $ABC \approx A'B'C'$, $\text{area } DEF = \text{area } D'E'F'$.

Ex. 25. The base of a triangular pyramid is an equilateral triangle, whose side is 16. Each lateral edge of the pyramid makes an angle of 60° with the base; find the altitude of the pyramid.

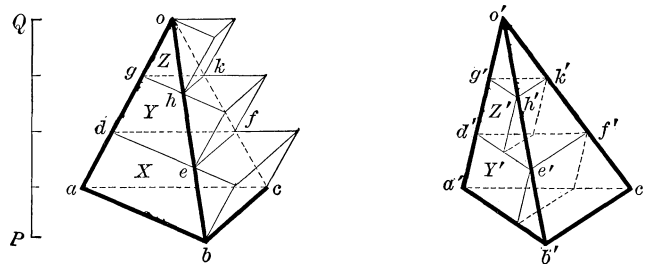
Ex. 26. The locus of all points equidistant from the faces of a tri-edral angle is the intersection of the planes which bisect its diedrals.

Ex. 27. Prove that the lateral area of a regular pyramid is greater than the area of its base.

Ex. 28. The base of a regular pyramid is a square whose side is 8. The slant height makes an angle 45° with the base; find the lateral area.

PROP. XIX. THEOREM

458. *Two triangular pyramids having equal altitudes and equivalent bases are equivalent.*



Given $o-abc$ and $o'-a'b'c'$ triangular pyramids with equal altitudes and $\approx$ bases.

To Prove $\text{vol. } o-abc = \text{vol. } o'-a'b'c'.$

Proof. 1. Place pyramids with bases in same plane; let PQ be common altitude, and divide it into three equal parts.

2. Through points of division pass planes $\parallel$ to base, cutting $o-abc$ in def , ghk , and $o'-a'b'c'$ in $d'e'f'$, $g'h'k'$; then

$$def \approx d'e'f', ghk \approx g'h'k'. \quad (\S 457)$$

3. With abc , def , ghk as *lower* bases, construct prisms X , Y , Z , with lateral edges $=$ and $\parallel$ to ad ; with $d'e'f'$, $g'h'k'$ as *upper* bases, construct prisms Y' , Z' , with lateral edges $=$ and $\parallel$ to $a'd'$; then prism $Y \approx Y'$, also $Z \approx Z'$. ($\S 442$)

4. Then, sum of prisms circumscribed about $o-abc$ exceeds sum of prisms inscribed in $o'-a'b'c'$ by vol. X .

5. $o-abc$ is $<$ sum of X , Y , Z ; and $>$ sum of prisms $\approx Y'$, Z' , with def , ghk as *upper* bases, and lateral edges $=$ and $\parallel$ to ad .

6. $o'-a'b'c'$ is $>$ ($Y' + Z'$); and $<$ sum of prisms $\approx X$, Y , Z , with $a'b'c'$, $d'e'f'$, $g'h'k'$ as *lower* bases, and lateral edges $=$ and $\parallel$ to $a'd'$.

7. Then volumes of pyramids differ by $<$ the difference of the volumes of the two systems of prisms; or by $<$ vol. X , by result (4).

8. By increasing number of divisions of PQ , vol. X can be made $<$ any assigned volume, however small.

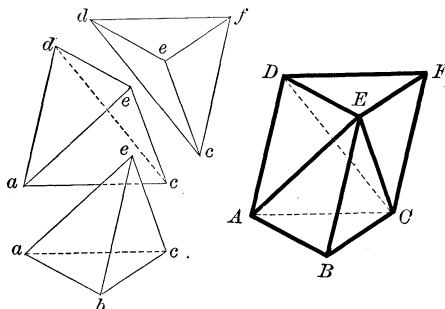
9. Then, volumes of pyramids cannot differ; or

$$\text{vol. } o-abc = \text{vol. } o'-a'b'c'.$$

459. Since vol. $o'-a'b'c'$ is $>$ (vol. $Y' + \text{vol. } Z'$), which = (vol. $Y + \text{vol. } Z$), and $<$ (vol. $X + \text{vol. } Y + \text{vol. } Z$), vol. $o'-a'b'c'$ must be $<$ vol. X ; and hence approaches the limit 0 when the number of subdivisions is indefinitely increased.

PROP. XX. THEOREM

460. *A triangular pyramid is equivalent to one-third of a triangular prism having the same base and altitude.*



Given triangular pyramid $E-ABC$, and prism $ABC-DEF$ with same base and altitude.

To Prove $\text{vol. } E-ABC = \frac{1}{3} \text{ vol. } ABC-DEF$.

Proof. 1. Passing plane through C, D, E , prism is divided into triangular pyramids $E-ABC, C-DEF, E-ACD$.

2. Since $E-ABC, C-DEF$ have base $ABC = DEF$ and same altitude (§§ 414, 374), volumes are equal (§ 458).

3. Since $C-DEF$ (or $E-DCF$), and $E-ACD$, have base $DCF = ACD$ (§ 105), and same altitude, volumes are equal.

(?)

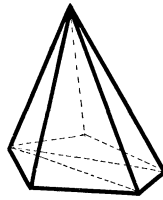
4. Since volumes of pyramids are equal,

$$\text{vol. } E-ABC = \frac{1}{3} \text{ vol. } ABC-DEF.$$

461. By § 460, the volume of a triangular pyramid is equal to one-third the product of its base and altitude. (§ 442)

PROP. XXI. THEOREM

462. The volume of any pyramid is equal to one-third the product of its base and altitude.



Given any pyramid.

To Prove its volume equal to one-third the product of its base and altitude.

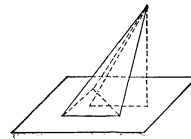
(Prove as in § 443.)

463. It follows from § 462 that

1. *Pyramids having equivalent bases and equal altitudes are equivalent.*
2. *Two pyramids having equal altitudes are to each other as their bases.*
3. *Two pyramids having equivalent bases are to each other as their altitudes.*
4. *Any two pyramids are to each other as the products of their bases by their altitudes.*

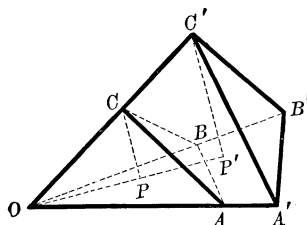
Ex. 29. The altitude of a pyramid is a , the base is b . A plane, parallel to the base, bisects one edge of the pyramid; find the volume of the pyramid cut off by the plane.

Ex. 30. The base of a pyramid is an equilateral triangle. The line from the vertex to the intersection of the altitudes of the triangular base makes an angle of 60° with the base of the pyramid, and the projection of the vertex of the pyramid on the base meets this plane at a point 16 from the intersection of the altitudes of the base. The altitude of the base is $8\sqrt{3}$; find the volume of the pyramid.



PROP. XXII. THEOREM

464. *Two tetraedrons having a triedral angle of one equal to a triedral angle of the other, are to each other as the edges including the equal triedral angles.*



Given V and V' the volumes of tetraedrons $O-ABC$ and $O-A'B'C'$, respectively, having common triedral $\angle O$.

To Prove
$$\frac{V}{V'} = \frac{OA \times OB \times OC}{OA' \times OB' \times OC'}.$$

Proof. 1. Draw lines $CP, C'P' \perp$ face $OA'B'$; let their plane intersect $OA'B'$ in line OPP' .

2. Since $OAB, OA'B'$ are the bases, and $CP, C'P'$ the altitudes, of pyramids $C-OAB, C'-OA'B'$,

$$\frac{V}{V'} = \frac{\text{area } OAB \times CP}{\text{area } OA'B' \times C'P'} (\S 463, 4) = \frac{\text{area } OAB}{\text{area } OA'B'} \times \frac{CP}{C'P'}. \quad (1)$$

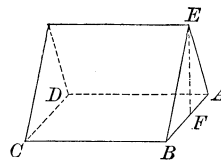
3. Now
$$\frac{\text{area } OAB}{\text{area } OA'B'} = \frac{OA \times OB}{OA' \times OB'}. \quad (\S 290)$$

4. In similar rt. $\triangle OCP, OC'P'$ (§§ 352, 237), $\frac{CP}{C'P'} = \frac{OC}{OC'}. (?)$

5. Substitute results (3) and (4) in (1).

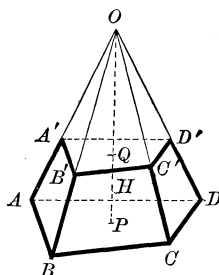
Ex. 31. The diagonals and edges of a cube determine triangles, which form the faces of six equal quadrangular pyramids.

Ex. 32. The volume of a right triangular prism is equal to the product of the area of a face, $ABCD$, and one-half the altitude, EF , of the base.



PROP. XXIII. THEOREM

465. *The volume of a frustum of a pyramid is equal to the sum of its bases and the mean proportional between its bases, multiplied by one-third its altitude.*



Given area of lower base B , area of upper base b , and altitude H , of AC' , a frustum of any pyramid $O-AC$.

To Prove $\text{vol. } AC' = (B + b + \sqrt{B \times b}) \times \frac{1}{3} H$ (§ 217).

Proof. 1. Let altitude OP cut $A'C'$ at Q .

2. Since AC' is difference of pyramids $O-AC$, $O-A'C'$,

$$\begin{aligned} \text{vol. } AC' &= B \times \frac{1}{3} (H + OQ) - b \times \frac{1}{3} OQ & (\S 462) \\ &= B \times \frac{1}{3} H + B \times \frac{1}{3} OQ - b \times \frac{1}{3} OQ \\ &= B \times \frac{1}{3} H + (B - b) \times \frac{1}{3} OQ. & (1) \end{aligned}$$

3. By § 456, $\frac{B}{b} = \frac{(H + OQ)^2}{OQ^2}$, or $\frac{\sqrt{B}}{\sqrt{b}} = \frac{H + OQ}{OQ}$.

4. Clear of fractions,

$$OQ \sqrt{B} = H \sqrt{b} + OQ \sqrt{b}, \text{ or } OQ (\sqrt{B} - \sqrt{b}) = H \sqrt{b}.$$

5. Multiply both sides by $\sqrt{B} + \sqrt{b}$,

$$OQ(B - b) = H \sqrt{B \times b} + H \times b.$$

6. Substitute this in (1).

Ex. 33. The altitude of a pyramid whose base is a square is 9; the area of the base is 400:

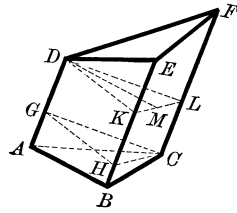
(a) find the lateral area;

(b) the volume;

(c) the distance from the vertex to the section parallel to the base and having an area 190.

PROP. XXIV. THEOREM

466. *The volume of a truncated triangular prism is equal to the product of a right section by one-third the sum of the lateral edges.*



Given GHC and DKL rt. sections of truncated triangular prism $ABC-DEF$.

To Prove

$$\text{vol. } ABC-DEF = \text{area } GHC \times \frac{1}{3} (AD + BE + CF). \quad (1)$$

Proof. 1. In rt. section DKL , draw $DM \perp KL$; truncated prism $ABC-DEF$ consists of rt. triangular prism $GHC-DKL$, and pyramids $D-EKLF$, $C-ABHG$.

2. Since lateral edges of a prism are equal (?),

$$\begin{aligned} \text{vol. } GHC-DKL &= \text{area } GHC \times GD \\ &= \text{area } GHC \times \frac{1}{3} (GD + HK + CL). \end{aligned} \quad (?)$$

3. Since DM is altitude of pyramid $D-EKLF$ (§ 387),

$$\text{vol. } D-EKLF = \text{area } EKLF \times \frac{1}{3} DM. \quad (?)$$

4. Since KL is altitude of trapezoid $EKLF$ (§ 350),

$$\begin{aligned} \text{vol. } D-EKLF &= \frac{1}{2} (KE + LF) \times KL \times \frac{1}{3} DM \\ &= (\frac{1}{2} KL \times DM) \times \frac{1}{3} (KE + LF) \\ &= \text{area } DKL \times \frac{1}{3} (KE + LF), \end{aligned} \quad (?)$$

$$\text{or} \quad \text{area } GHC \times \frac{1}{3} (KE + LF). \quad (?)$$

5. Similarly, $\text{vol. } C-ABHG = \text{area } GHC \times \frac{1}{3} (AG + BH)$.

6. Adding results (2), (4), (5), vol. $A-BCDEF$ equals area $GHC \times \frac{1}{3}(AG + GD + BH + HK + KE + CL + LF)$.

467. By § 466, the volume of a truncated right triangular prism is equal to the product of its base by one-third the sum of the lateral edges.

Ex. 34. Given any tetraedron and the medians of the faces which form a triedral, and planes passed through these medians in pairs, then these planes and the base form a second tetraedron which is equivalent to one-fourth the given tetraedron.

Ex. 35. A side of the lower base of a frustum of a pyramid whose base is a square is 12 feet, a side of the upper base is 8 feet; the altitude of the frustum is 18 feet; find its volume.

Ex. 36. A pyramid has a triangular base whose area is 144 square feet; its altitude is 14 feet. At what distance from the vertex must a plane be passed parallel to the base that the area of the section made by the plane may be 96 square feet?

Ex. 37. A frustum of a pyramid is cut by a plane, parallel to and halfway between the bases; find the ratio of the perimeter of this section of the frustum to the sum of the perimeters of the bases. Do the areas of the bases and the section have the same ratio as their perimeters?

Ex. 38. A concrete stack is in the form of a frustum of a rectangular pyramid. The lower base is 10 feet long and 8 feet wide, the upper base 5 feet long and 4 feet wide, the height, 50 feet—all outside measurements. The thickness of the wall is one foot. How much Portland cement was needed for construction, if the mixture is one part cement to nine parts sand and gravel?

Ex. 39. The altitude of a triangular pyramid is 14. The sides of the base are 6, 8, and 10; find the volume of the frustum formed by passing a plane through the pyramid parallel to the base and 7 from the vertex.

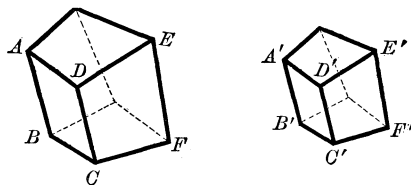
Ex. 40. The face angles at the vertex of a regular triangular pyramid are right angles. Find an expression for the volume in terms of an edge.

SIMILAR POLYEDRONS

468. Def. Two polyhedrons are said to be *similar* when they have the same number of faces similar each to each and similarly placed, and have their homologous polyedral angles equal.

PROP. XXV. THEOREM

469. *The ratio of any two homologous edges of two similar polyhedrons is equal to the ratio of any other two homologous edges.*



Given, in similar polyhedrons AF and $A'F'$, edge AB homologous to edge $A'B'$, and edge EF to edge $E'F'$; and faces AC and DF similar to faces $A'C'$ and $D'F'$, respectively.

To Prove $\frac{AB}{A'B'} = \frac{EF}{E'F'}$.

(By § 234, 2, $\frac{AB}{A'B'} = \frac{CD}{C'D'}$.)

470. It follows from § 469 that *any two homologous faces of two similar polyhedrons are to each other as the squares of any two homologous edges.*

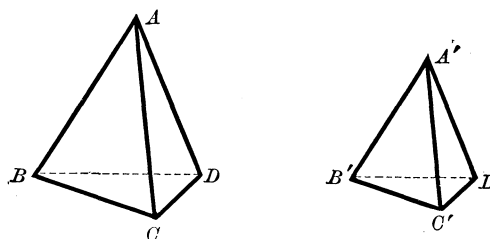
For by § 291, $\frac{\text{area } ABCD}{\text{area } A'B'C'D'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{EF}^2}{\overline{E'F'}^2}$.

471. By § 470, *the entire surfaces of two similar polyhedrons are to each other as the squares of any two homologous edges.*

For by § 224, $\frac{\text{area } ABCD + \text{area } CDEF, \text{ etc.}}{\text{area } A'B'C'D' + \text{area } C'D'E'F', \text{ etc.}} = \frac{\overline{EF}^2}{\overline{E'F'}^2}$.

PROP. XXVI. THEOREM

472. *Two tetraedrons are similar when the faces including a triedral angle of one are similar, respectively, to the faces including a triedral angle of the other, and similarly placed.*



Given, in tetraedrons $ABCD$, $A'B'C'D'$, face ABC similar to $A'B'C'$, ACD to $A'C'D'$, and ADB to $A'D'B'$.

To Prove $ABCD$, $A'B'C'D'$ similar.

Proof. 1. We have $\frac{BC}{B'C'} = \frac{AC}{A'C'} = \frac{CD}{C'D'} = \frac{AD}{A'D'} = \frac{BD}{B'D'}$. (?)

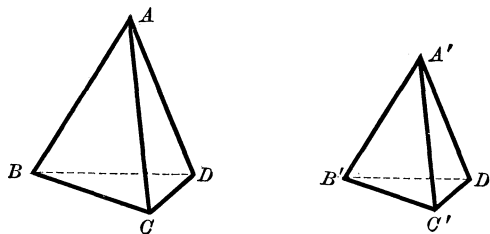
2. Then BCD , $B'C'D'$ are similar; and $\angle BAC$, CAD , DAB equal, respectively, to $\angle B'A'C'$, $C'A'D'$, $D'A'B'$. (?)

3. Then triedral $\angle A-BCD$, $A'-B'C'D'$ are equal. (§ 409, 1)

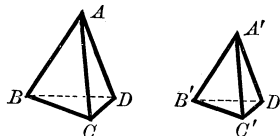
4. Similarly, any two homologous triedrals are equal; and $ABCD$, $A'B'C'D'$ similar. (?)

PROP. XXVII. THEOREM

473. *Two tetraedrons are similar when a dihedral angle of one is equal to a dihedral angle of the other, and the faces including the equal dihedral angles similar each to each, and similarly placed.*



Given, in tetraedrons $ABCD$, $A'B'C'D'$, diedral $\angle AB$ equal to diedral $\angle A'B'$, face ABC similar to $A'B'C'$, and face ABD similar to $A'B'D'$.



To Prove $ABCD$, $A'B'C'D'$ similar.

Proof. 1. Apply $A'B'C'D'$ to $ABCD$ so that diedral $A'B'$ shall coincide with diedral AB , and point A' with A .

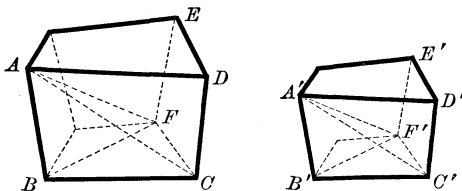
2. Since $\angle B'A'C' = \angle BAC$ and $\angle B'A'D' = \angle BAD$, $A'C'$ will coincide with AC , and $A'D'$ with AD , and
 $\angle C'A'D' = \angle CAD$.

3. Since $\frac{A'C'}{AC} = \frac{A'B'}{AB} = \frac{A'D'}{AD}$ (?), $\triangle C'A'D'$, CAD are similar.
 (§ 242)

4. Then $ABCD$, $A'B'C'D'$ are similar by § 472.

PROP. XXVIII. THEOREM

474. *Two similar polyedrons may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed.*



Given AF and $A'F'$ similar polyedrons, vertices A and A' being homologous.

To Prove that they may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed.

Proof. 1. Divide all faces of AF , except the ones having A as a vertex, into $\triangle$, by drawing diagonals from *one vertex*.

2. Divide all faces of $A'F'$, except those having A' as a vertex, into $\triangle$ similar to those in AF , and similarly placed. (§ 247)

3. Passing planes through A and A' and the sides of the $\triangle$, AF and $A'F'$ are decomposed into same number of tetraedrons, similarly placed.

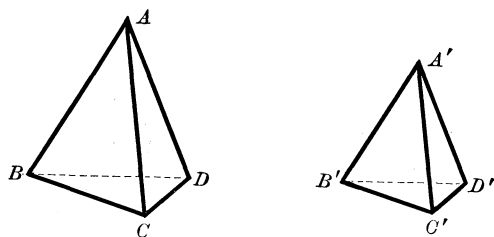
4. If $ABCF$, $A'B'C'F'$ are homologous tetraedrons, $\triangle ABC$ is similar to $A'B'C'$, and BCF to $B'C'F'$. (§ 247)

5. Since AF and $A'F'$ are similar, diedrals BC and $B'C'$ are equal (?); then $ABCF$, $A'B'C'F'$ are similar. (§ 473)

6. Similarly, we may prove any two homologous tetraedrons similar.

PROP. XXIX. THEOREM

475. *Two similar tetraedrons are to each other as the cubes of their homologous edges.*



Given V , V' the volumes of similar tetraedrons $ABCD$, $A'B'C'D'$, vertices A , A' being homologous.

To Prove
$$\frac{V}{V'} = \frac{AB^3}{A'B'^3}.$$

Proof. 1. Since diedrals at A , A' are equal (?),

$$\frac{V}{V'} = \frac{AB \times AC \times AD}{A'B' \times A'C' \times A'D'} \quad (\S 464) = \frac{AB}{A'B'} \times \frac{AC}{A'C'} \times \frac{AD}{A'D'}. \quad (1)$$

2. Express $\frac{AC}{A'C'}$ and $\frac{AD}{A'D'}$ in terms of AB , $A'B'$. (§ 469)

3. Substitute this in (1).

476. *Any two similar polyhedrons are to each other as the cubes of their homologous edges.*

For any two similar polyhedrons may be decomposed into the same number of tetraedrons, similar each to each (§ 474); and any two homologous tetraedrons are to each other as the cubes of their homologous edges. (§ 475)

REGULAR POLYEDRONS

477. Def. A *regular polyedron* is a polyedron whose faces are equal regular polygons, and whose polyedral angles are all equal.

PROP. XXX. THEOREM

478. *It is not possible to construct more than five regular convex polyedrons.*

Proof. 1. A convex polyedral $\angle$ must have at least three faces, and the sum of its face $\angle$'s $< 360^\circ$. (§ 407)

2. Since the $\angle$ of an equilateral $\triangle$ is 60° , we can form convex polyedral $\angle$'s by combining 3, 4, or 5 equilateral $\triangle$'s, but not more than 5; then not more than 3 regular convex polyedrons can be bounded by equilateral $\triangle$'s.

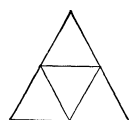
3. Since the $\angle$ of a square is 90° , we can form a convex polyedral $\angle$ by combining 3 squares, but not more than 3; then not more than one regular convex polyedron can be bounded by squares.

4. Since the $\angle$ of a regular pentagon is 108° , we can form a convex polyedral $\angle$ by combining 3 regular pentagons, but not more than 3; then not more than one regular convex polyedron can be bounded by regular pentagons.

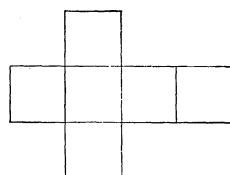
5. Since the $\angle$ of a regular hexagon is 120° , and of a regular polygon of more than 6 sides $> 120^\circ$, no convex polyedral $\angle$ can be formed by combining regular hexagons, or regular polygons of more than 6 sides.

479. Note. The regular polyedrons are the *regular tetraedron*, *octaedron*, and *icosaedron*, bounded respectively by four, eight, and twenty equilateral triangles; the *regular hexaedron*, or *cube*, bounded by six squares; and the *regular dodecaedron*, bounded by twelve regular pentagons.

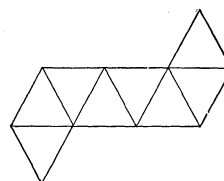
To construct them, draw the following figures on cardboard; cut them out entire, and on the interior lines cut the cardboard half through; the edges may then be brought together to form the respective solids.



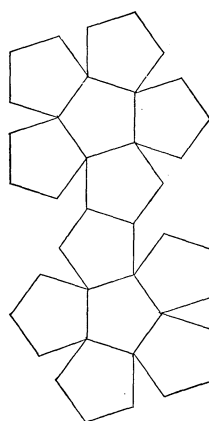
Tetraedron



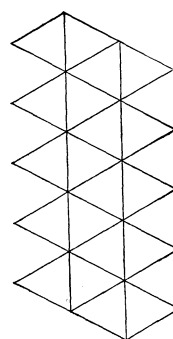
Hexaedron



Octaedron



Dodecaedron



Icosaedron

Ex. 41. One edge of a regular tetraedron is a ; find the volume.

Ex. 42. In any regular tetraedron planes passed through the bisectors, in pairs, of the face angles of any triedral angle, enclose, with the base, a tetraedron whose volume is $\frac{1}{4}$ that of the given tetraedron.

Ex. 43. One edge of a regular tetraedron is 6: find the area of the entire surface and the volume.

Ex. 44. Find the volume of a regular tetraedron whose edge is one foot. Compare this volume with that of a cube having the same edge.

Ex. 45. In each face of a regular tetraedron draw lines joining the middle points of the edges. Do the triangles thus formed form the faces of a regular tetraedron?

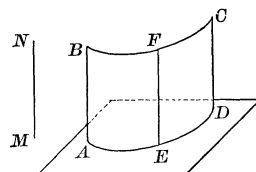
BOOK VIII

THE CYLINDER AND CONE

DEFINITIONS

480. A *cylindrical surface* is a surface generated by a moving straight line, which constantly intersects a given plane curve, and in all its positions is parallel to a given straight line, not in the plane of the curve.

Thus, if line AB moves so as constantly to intersect plane curve AD , and is constantly $\parallel$ line MN , not in the plane of the curve, it generates a cylindrical surface.



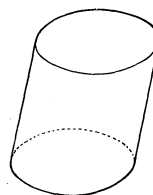
We call the curve the *directrix*, the moving line the *generatrix*, and any position of it, EF , an *element* of the surface.

A *cylinder* is a solid bounded by a cylindrical surface, called the *lateral surface*, and two parallel planes called the *bases*.

The *altitude* of a cylinder is the perpendicular distance between the planes of its bases.

A *right cylinder* is a cylinder the elements of whose lateral surface are perpendicular to its bases.

A *circular cylinder* is a cylinder whose base is a circle.



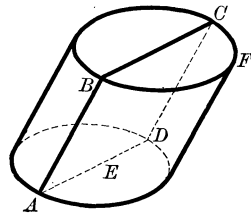
A right circular cylinder is called a *cylinder of revolution* because it may be generated by the revolution of a rectangle about one of its sides as an axis (§ 371).

Similar cylinders of revolution are cylinders by the revolution of similar rectangles about homologous sides as axes.

481. It follows from the definitions of § 480 that *the elements of the lateral surface of a cylinder are equal and parallel.* (§ 365)

PROP. I. THEOREM

482. *A section of a cylinder made by a plane passing through an element of the lateral surface is a parallelogram.*



Given $ABCD$ a section of cylinder AF , made by a plane passing through AB , an element of the lateral surface.

To Prove section $ABCD$ a $\square$.

Proof. 1. We have $AD, BC \parallel$ str. lines. (§§ 348, 364)

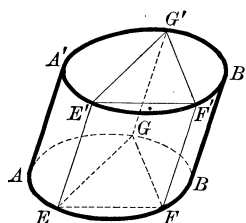
2. If a str. line be drawn from C in plane $AC \parallel AB$, it is an element. (§§ 481, 71)

3. Then, it must be the intersection of plane AC and the cylindrical surface and coincide with CD ; and $CD \parallel AB$.

483. It follows from § 482 that *a section of a right cylinder made by a plane perpendicular to its base is a rectangle.*

PROP. II. THEOREM

484. *The bases of a cylinder are equal.*



Given cylinder AB' .

To Prove base $A'B' = \text{base } AB$.

Proof. 1. Take E', F', G' any three points in perimeter $A'B'$; draw elements $E'E, F'F, G'G$, and lines $EF, FG, GE, E'F', F'G', G'E'$.

2. We have EE', FF' equal and $\parallel$ (?); whence $EE'F'F$ is a $\square$. (?)

3. Then $E'F' = EF$. (?)

Similarly $E'G' = EG$,

$$F'G' = FG.$$

4. Then $\triangle E'F'G' = \triangle EFG$ (?); and base $A'B'$ may be applied to AB so that E' shall fall at E , F' at F , G' at G .

5. Then, *all* points in perimeter $A'B'$ will fall in perimeter AB , and bases are equal.

The following are consequences of § 484.

485. *The sections of a cylinder made by two parallel planes cutting all the elements are equal.*

For they are the bases of a cylinder.

486. *The section of a cylinder made by a plane parallel to the base is equal to the base.*

MEASUREMENT OF CYLINDERS

DEFINITIONS

487. The *lateral area* of a cylinder is the area of its lateral surface.

A *right section* of a cylinder is a section made by a plane perpendicular to the elements of its lateral surface.

A prism is said to be *inscribed in a cylinder* when its lateral edges are elements of the cylindrical surface, and its bases inscribed in the bases of the cylinder.

A plane is said to be *tangent* to a cylinder when it contains one, and only one, element of the lateral surface.

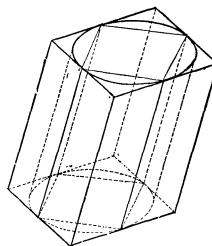
A prism is said to be *circumscribed about a cylinder* when its lateral faces are tangent to the cylinder, and its bases circumscribed about the bases of the cylinder.

488. If a prism whose base is a regular polygon be inscribed in, or circumscribed about, a circular cylinder (§ 480), and the number of sides of its base be indefinitely increased,

1. The lateral area of the prism approaches the lateral area of the cylinder as a limit.

2. The volume of the prism approaches the volume of the cylinder as a limit.

3. The perimeter of a right section of the prism approaches the perimeter of a right section of the cylinder as a limit.*



Ex. 1. Find area of the section made by a plane drawn through an element of the lateral surface of a cylinder of revolution, whose altitude is 12, and radius of base 6, if the distance from the centre of the base to the cutting plane is 3.

* Rigorous proofs of these statements are beyond the scope of the present treatise.

PROP. III. THEOREM

489. *The lateral area of a circular cylinder is equal to the perimeter of a right section multiplied by an element of the lateral surface.*

Given S the lateral area, P the perimeter of a rt. section, E an element of the lateral surface, of a circular cylinder.

To Prove $S = P \times E.$ (1)

Proof. 1. Inscribe in the cylinder a prism whose base is a regular polygon; let S' denote its lateral area, and P' the perimeter of a rt. section.

2. We have $S' = P' \times E.$ (§ 429)

3. If number of sides of base be indefinitely increased, find limits of S' and $P' \times E.$ (§ 488)

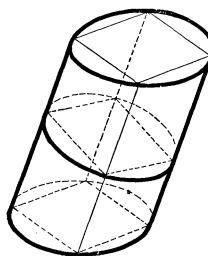
4. We have (1) by Theorem of Limits. (§ 187)

490. By § 489, *the lateral area of a cylinder of revolution is equal to the circumference of its base multiplied by its altitude.*

491. If S denotes the lateral area, T the total area, H the altitude, R the radius of the base, of a cylinder of revolution,

$$S = 2\pi RH. \quad (\S\ 334)$$

$$\text{And, } T = 2\pi RH + 2\pi R^2 (\S\ 337) = 2\pi R (H + R).$$



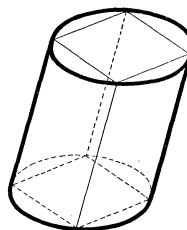
PROP. IV. THEOREM

492. *The volume of a circular cylinder is equal to the product of its base and altitude.*

Given V the volume, B the area of the base, and H the altitude, of a circular cylinder.

To Prove $V = B \times H.$ (1)

Proof. 1. Inscribe in cylinder a prism whose base is a regular polygon; let V' denote its volume, B' the area of its base.



2. We have $V' = B' \times H$. (§ 443)
3. If number of sides of base of prism be indefinitely increased, find limits V' and $B' \times H$. (§§ 488, 329)
4. Apply Theorem of Limits.

493. If V denotes the volume, H the altitude, and R the radius of the base, of a circular cylinder,

$$V = \pi R^2 H. \quad (?)$$

Ex. 2. The radius of the base of a right circular cylinder is 9. The altitude is 8. Find the lateral area. If this cylinder were oblique, would a right section be a circle?

Ex. 3. The lateral area of a cylinder of revolution is 120π . The area of the base is 36π . Find the altitude.

Ex. 4. A point in space moves so that it is always at a constant distance from a fixed straight line; find its locus.

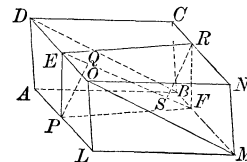
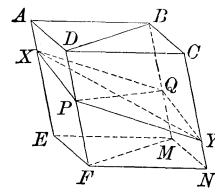
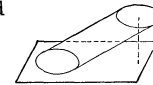
Ex. 5. The radius of the base of a cylinder of revolution is 12. The altitude is 13. Find the volume.

Ex. 6. The radius of the base of an oblique circular cylinder is 6. The distance between the planes of the bases is 10; find the volume.

Ex. 7. The radius of the base of a circular cylinder is 8. The elements of the lateral surface make an angle of 30° with the plane of the base. The length of an element is 25. Find volume of cylinder.

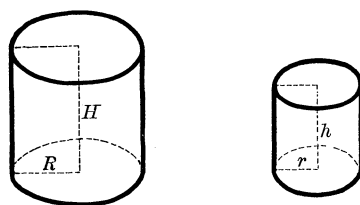
Ex. 8. The line which joins any two opposite points in the diagonally opposite edges of a parallelepiped is bisected by the plane passed through the other two edges.

Ex. 9. If from any two points in the diagonally opposite base edges of a parallelepiped perpendiculars be drawn to the plane passed through the other two edges, these perpendiculars are equal.



PROP. V. THEOREM

494. *The lateral or total areas of two similar cylinders of revolution (§ 480) are to each other as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*



Given S and s the lateral areas, T and t the total areas, V and v the volumes, H and h the altitudes, and R and r the radii of the bases, of two similar cylinders of revolution.

To Prove $\frac{S}{s} = \frac{T}{t} = \frac{H^2}{h^2} = \frac{R^2}{r^2}$, and $\frac{V}{v} = \frac{H^3}{h^3} = \frac{R^3}{r^3}$.

Proof. 1. Since rectangles are similar, $\frac{H}{h} = \frac{R}{r} = \frac{H+R}{h+r}$. (?)

2. By § 491, $\frac{S}{s} = \frac{2\pi RH}{2\pi rh} = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{H^2}{h^2}$,

and $\frac{T}{t} = \frac{2\pi R(H+R)}{2\pi r(h+r)} = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{H^2}{h^2}$.

3. By § 493, $\frac{V}{v} = \frac{\pi R^2 H}{\pi r^2 h} = \frac{R^2}{r^2} \times \frac{R}{r} = \frac{R^3}{r^3} = \frac{H^3}{h^3}$.

Ex. 10. Water weighs 1000 Kg. per cubic meter. The average weight of the same amount of milk is 1033 Kg. A cylindrical milk-can contains $8\frac{1}{2}$ gallons of milk. How many pounds of milk will it contain? If the diameter of its base is 10 inches, what is its altitude?

Ex. 11. An oil tank is a cylinder of revolution, whose length is 30 feet, and radius of base 3.5 feet. The average weight of petroleum is 795 Kg. per cubic meter. What weight of oil will the tank contain?

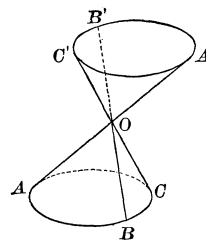
THE CONE

DEFINITIONS

495. A *conical surface* is a surface generated by a moving straight line, which constantly intersects a given plane curve, and passes through a given point not in the plane of the curve.

Thus, if line OA moves so as constantly to intersect plane curve ABC , and constantly passes through point O , not in the plane of the curve, it generates a conical surface.

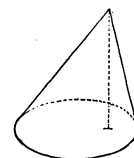
We call the moving line the *generatrix*, the curve the *directrix*, the given point the *vertex*, and any position of the generatrix, OB , an *element* of the surface.



If the generatrix be supposed indefinite in length, it will generate two conical surfaces of indefinite extent, $O-A'B'C'$, $O-ABC$, called the *upper* and *lower nappes*, respectively.

A *cone* is a solid bounded by a conical surface, and a plane cutting all its elements.

We call the plane the *base* of the cone, and the conical surface the *lateral surface*; the *altitude* is the perpendicular distance from the vertex to the plane of the base.



A *circular cone* is a cone whose base is a circle.

The *axis* of a circular cone is a straight line drawn from the vertex to the centre of the base.

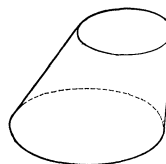
A *right circular cone* is a circular cone whose axis is perpendicular to its base.

A right circular cone is called a *cone of revolution* because it may be generated by the revolution of a right triangle about one of its legs as an axis.

Similar cones of revolution are generated by the revolution of similar right triangles about homologous legs as axes.

A *frustum of a cone* is a portion of a cone included between the base and a plane parallel to the base.

The base of the cone is called the *lower base*, and the section made by the plane the *upper base*, of the frustum.

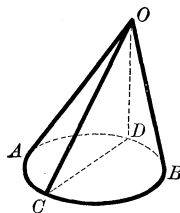


The *altitude* is the perpendicular distance between the planes of the bases.

A plane is said to be *tangent* to a cone, or frustum of a cone, when it contains one, and only one, element of the lateral surface.

PROP. VI. THEOREM

496. *A section of a cone made by a plane passing through the vertex is a triangle.*



Given OCD a section of cone OAB made by a plane passing through vertex O .

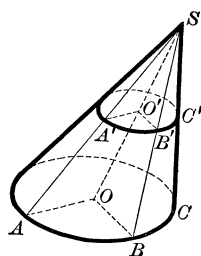
To Prove section OCD a $\triangle$.

Proof. 1. We have CD a str. line. (?)

2. Str. lines OC , OD , drawn in plane OCD , are elements (§ 495), and hence the intersections of the plane with the conical surface; then section OCD is bounded by str. lines.

PROP. VII. THEOREM

497. *A section of a circular cone made by a plane parallel to the base is a circle.*



Given $A'B'C'$ a section of circular cone $S-ABC$, made by a plane $\parallel$ to the base.

To Prove $A'B'C'$ a $\odot$.

Proof. 1. Take A', B' any two points in perimeter $A'B'C'$.

2. Let planes determined by A', B' , and axis OS intersect base in radii OA, OB , the section in $O'A', O'B'$, and the lateral surface in str. lines $SA'A, SB'B$. (§ 496)

3. Then $O'A' \parallel OA$ and $O'B' \parallel OB$ (?), and $\triangle SOA, SOB$ are similar to $\triangle SO'A', SO'B'$, respectively. (?)

4. Then, $\frac{O'A'}{OA} = \frac{SO'}{SO}$ and $\frac{O'B'}{OB} = \frac{SO'}{SO}$. (?)

Then, $\frac{O'A'}{OA} = \frac{O'B'}{OB}$. (?)

5. Since $OA = OB$ (?) $O'A' = O'B'$; A', B' being any two points in perimeter, section $A'B'C'$ is a $\odot$.

Ex. 12. Two circular cylinders have equal altitudes, but the radius of the base of the one is double the radius of the base of the other. Compare (a) their lateral areas; (b) their volumes.

Ex. 13. Two circular cylinders have equal altitudes, but the radius of the base of the first is $\frac{1}{3}$ the radius of the base of the second. Compare (a) their lateral areas; (b) their volumes.

MEASUREMENT OF CONES

DEFINITIONS

498. The *lateral area* of a cone, or frustum of a cone, is the area of its lateral surface.

The *slant height* of a cone of revolution is the straight line from the vertex to any point in the circumference of the base.

The *slant height* of a frustum of a cone of revolution is that portion of the slant height of the cone included between the bases of the frustum.

499. A pyramid is said to be *inscribed in a cone* when its vertex coincides with the vertex of the cone, and its base is inscribed in the base of the cone.

A pyramid is said to be *circumscribed about a cone* when its vertex coincides with the vertex of the cone, and its base is circumscribed about the base of the cone.

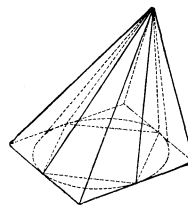
A frustum of a pyramid is said to be *inscribed in a frustum of a cone* when its bases are inscribed in the bases of the frustum of the cone.

A frustum of a pyramid is said to be *circumscribed about a frustum of a cone* when its bases are circumscribed about the bases of the frustum of the cone.

500. If a pyramid whose base is a regular polygon be inscribed in, or circumscribed about, a circular cone (§ 495), and the number of sides of its base be indefinitely increased,

1. The lateral area of the pyramid approaches the lateral area of the cone as a limit.

2. The volume of the pyramid approaches the volume of the cone as a limit.*



* Rigorous proofs of these statements are beyond the scope of the present treatise.

501. *If a frustum of a pyramid whose base is a regular polygon be inscribed in, or circumscribed about, a frustum of a circular cone, and the number of sides of its base be indefinitely increased,*

1. *The lateral area of the frustum of the pyramid approaches the lateral area of the frustum of the cone as a limit.*

2. *The volume of the frustum of the pyramid approaches the volume of the frustum of the cone as a limit.**

PROP. VIII. THEOREM

502. *The lateral area of a cone of revolution is equal to the circumference of its base, multiplied by one-half its slant height.*

Given S the lateral area, C the circumference of the base, and L the slant height, of a cone of revolution.

To Prove $S = C \times \frac{1}{2} L$.

Proof. 1. Circumscribe about the cone a regular pyramid; let S' denote its lateral area, and C' the perimeter of its base.

2. Since sides of base of pyramid are bisected at points of contact with base of cone (§ 174), the slant heights of pyramid and cone are equal, and $S' = C' \times \frac{1}{2} L$. (§§ 450, 453)

3. If number of sides of base of pyramid be indefinitely increased, find limits S' and $C' \times \frac{1}{2} L$. (§§ 500, 329)

4. Apply Theorem of Limits.

503. If S denotes lateral area, T total area, L the slant height, and R the radius of the base, of a cone of revolution,

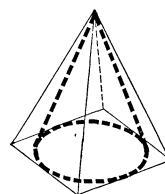
$$S = 2\pi R \times \frac{1}{2} L (?) = \pi RL.$$

And,
$$T = \pi RL + \pi R^2 (?) = \pi R(L + R).$$

Ex. 14. The diameter of the base of a right circular cone is equal to the altitude. Find the lateral and total areas.

Ex. 15. The edge of a regular tetraedron is 8: find area of the lateral surface of the circumscribed cone.

* Rigorous proofs of these statements are beyond the scope of the present treatise.



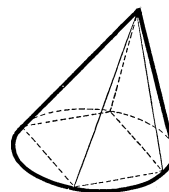
PROP. IX. THEOREM

504. *The volume of a circular cone is equal to the area of its base, multiplied by one-third its altitude.*

Given V the volume, B the area of the base, and H the altitude, of a circular cone.

To Prove $V = B \times \frac{1}{3} H$.

(Inscribe a pyramid whose base is a regular polygon.)

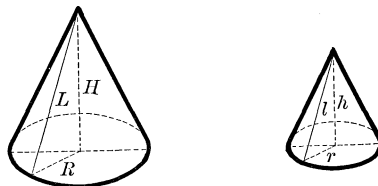


505. If V denotes the volume, H the altitude, and R the radius of the base, of a circular cone,

$$V = \frac{1}{3} \pi R^2 H. \quad (?)$$

PROP. X. THEOREM

506. *The lateral or total areas of two similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their slant heights, or as the cubes of their altitudes, or as the cubes of the radii of their bases.*



Given S and s the lateral areas, T and t the total areas, V and v the volumes, L and l the slant heights, H and h the altitudes, and R and r the radii of the bases, of two similar cones of revolution (§ 495).

To Prove $\frac{S}{s} = \frac{T}{t} = \frac{L^2}{l^2} = \frac{H^2}{h^2} = \frac{R^2}{r^2}$, and $\frac{V}{v} = \frac{L^3}{l^3} = \frac{H^3}{h^3} = \frac{R^3}{r^3}$.

(The proof is left to the pupil; compare § 494.)

PROP. XI. THEOREM

507. *The lateral area of a frustum of a cone of revolution is equal to the sum of the circumferences of its bases, multiplied by one-half its slant height.*

Given S the lateral area, C, c the circumferences of the bases, L the slant height, of a frustum of a cone of revolution.

To Prove $S = (C + c) \times \frac{1}{2} L$.

Proof. 1. Circumscribe about the frustum of the cone a frustum of a regular pyramid; let S' denote its lateral area. C', c' the perimeters of its bases.

2. As in § 502, the slant heights of the frustums are equal.

3. Then, $S' = (C' + c') \times \frac{1}{2} L$. (§ 454)

4. If number of sides of bases of frustum of pyramid be indefinitely increased, find limits S' and $(C' + c') \times \frac{1}{2} L$. (§§ 501, 329)

5. Apply Theorem of Limits.

508. If S denotes the lateral area, L the slant height, R, r the radii of the bases, of a frustum of a cone of revolution,

$$S = (2\pi R + 2\pi r) \times \frac{1}{2} L (?) = \pi(R + r) L.$$

509. We may write the first result of § 508

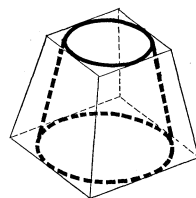
$$S = 2\pi \times \frac{1}{2} (R + r) \times L.$$

But, $2\pi \times \frac{1}{2} (R + r)$ is the circumference of a section equally distant from the bases. (§ 132)

Whence, *the lateral area of a frustum of a cone of revolution is equal to the circumference of a section equally distant from its bases, multiplied by its slant height.*

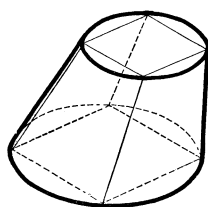
Ex. 16. In two similar cones of revolution, the slant height of the first is 16 and its lateral area is 128π . The lateral area of the second is 50π : find the radius of the base of the second.

Ex. 17. The area of the entire surface of a cylinder of revolution is 96π . Its altitude and diameter are equal. Find radius of base.



PROP. XII. THEOREM

510. *The volume of a frustum of a circular cone is equal to the sum of its bases and a mean proportional between its bases, multiplied by one-third its altitude.*



Given V the volume, B , b the areas of the bases, H the altitude, of a frustum of a circular cone.

To Prove $V = (B + b + \sqrt{B \times b}) \times \frac{1}{3} H.$

(Inscribe a frustum of a pyramid whose base is a regular polygon; then apply § 465.)

511. If V denotes the volume, H the altitude, and R and r the radii of the bases, of a frustum of a circular cone,

$$B = \pi R^2, b = \pi r^2, \text{ and } \sqrt{B \times b} = \sqrt{\pi^2 R^2 r^2} = \pi Rr. \quad (?)$$

Then, $V = (\pi R^2 + \pi r^2 + \pi Rr) \times \frac{1}{3} H = \frac{1}{3} \pi (R^2 + r^2 + Rr) H.$

Ex. 18. The radius of the base of a circular cylinder is a feet. The distance between the bases is equal to the distance from the centre of the lower base to the projection of the centre of the upper base upon the plane of the lower base: find the volume. Is there more than one such cylinder?

Ex. 19. The radius of the base of a cone is 8. A line joining the vertex and the centre of the base makes an angle of 30° with the base. The distance from the centre of the base to the projection of the vertex on the plane of the base is 12. Find volume of cone.

Ex. 20. In Ex. 19, if a plane be passed through the middle point of

the altitude and parallel to the base of the cone, find the volume of the frustum formed.

Ex. 21. The lateral area of a right circular cone is $16\sqrt{5}\pi$. The altitude of the cone and the diameter of its base are equal. Find the area of the entire surface.

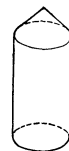
Ex. 22. The volumes of two similar polyhedrons are 27 and 512 cubic feet, respectively. Compare their homologous edges. If the area of the surface of the first is 36 square feet, find the area of the surface of the second.

Ex. 23. If four straight lines not in the same plane enclose a warped surface, the lines which join the middle points of these lines in order enclose a parallelogram.

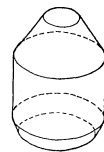
Ex. 24. A portion of the Chicago Drainage Canal is in the form of a right prism, whose base is an isosceles trapezoid. Width at water-line 162 feet. Width at bottom 160 feet. Depth of water 22 feet. In one minute 300,000 cubic feet of water pass a given point. Find the rate per hour of the current.

Ex. 25. Three metal pipes are each 30 inches long. The first two are in the form of a right prism, with a square and a regular hexagon, respectively, as bases; the third in the form of a cylinder of revolution, the radius of whose base is 10. The bases of the three pipes have equal perimeters. Find their volumes.

Ex. 26. A silo is in the form of a cylinder of revolution 18 feet in altitude surmounted by a roof in the form of a conical surface whose elements are inclined 45° to the upper base of the cylinder. The diameter of the base of the cylinder is 10 feet. Find the number of square feet of tin necessary to roof the silo, also the volume of ensilage the cylindrical part will contain.



Ex. 27. A cistern made of concrete is in the form of a cylinder of revolution with the frustums of two cones of revolution. Diameter of base of cylinder 12 feet; altitude of cylinder 8 feet; altitude of lower frustum 2 feet; diameter of bottom of cistern 10 feet; altitude of upper frustum 5 feet; diameter of opening 5 feet — all outside measurements. Inside diameter of base of cylinder, 8 feet. Inside diameter of upper base of lower frustum 8 feet, of bottom 6 feet. Inside diameter of lower base of upper frustum 8 feet, of upper base 3 feet. Using one part cement to nine parts sand and gravel, find amount of cement necessary for construction.



BOOK IX

THE SPHERE

DEFINITIONS

512. A *sphere* is a solid bounded by a surface, all points of which are equally distant from a point within called the *centre*.

A *radius* of a sphere is a straight line drawn from the centre to the surface; a *diameter* is a straight line drawn through the centre, having its extremities in the surface.

513. By the definition of § 512, *all radii of a sphere are equal*.

Also, all diameters are equal, since each is the sum of two radii.

514. *Two spheres are equal when their radii are equal.*

For they can be applied one to the other so that their surfaces shall coincide throughout.

Conversely, *the radii of equal spheres are equal.*

515. A line (or plane) is said to be *tangent to a sphere* when it has one, and only one, point in common with the surface; the common point is called the *point of contact*.

A polyedron is said to be *inscribed in a sphere* when all its vertices lie in the surface of the sphere; in this case the sphere is said to be *circumscribed about the polyedron*.

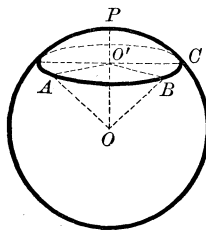
A polyedron is said to be *circumscribed about a sphere* when all its faces are tangent to the sphere; in this case the sphere is said to be *inscribed in the polyedron*.

516. *A sphere may be generated by the revolution of a semi-circle about its diameter as an axis.*

For all points of such a surface are equally distant from the centre of the $\odot$. (?)

PROP. I. THEOREM

517. *A section of a sphere made by a plane is a circle.*



Given ABC a section of sphere APC made by a plane.

To Prove ABC a $\odot$.

Proof. 1. If O is centre of sphere, draw $OO' \perp$ plane ABC .

2. Take A, B any two points in perimeter ABC ; draw lines $OA, OB, O'A, O'B$.

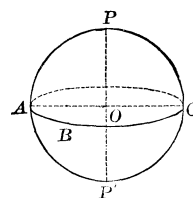
3. Since $OA = OB$ (?), $O'A = O'B$. (§ 357)

4. Since A, B are *any* two points in perimeter, ABC is a $\odot$.

518. Defs. A *great circle* of a sphere is a section made by a plane passing through the centre; as ABC .

A *small circle* is a section made by a plane which does not pass through the centre.

The *diameter* perpendicular to a circle of a sphere is called the *axis* of the circle; its extremities are called the *poles*.



The following are immediate consequences of §§ 517, 518.

519. *The axis of a circle of a sphere passes through the centre of the circle.*

520. *All great circles of a sphere are equal.*

For their radii are radii of the sphere.

521. *Every great circle bisects the sphere and its surface.*

For if the portions of the sphere formed by the plane of the great $\odot$ be separated, and placed so that their plane surfaces coincide, the spherical surfaces falling on the same side of this plane, the two spherical surfaces will coincide throughout; for all points of either surface are equally distant from the centre.

522. *Any two great circles bisect each other.*

For the intersection of their planes is a diameter of the sphere, and therefore a diameter of each $\odot$. (§ 153)

523. *Between any two points on the surface of a sphere, not the extremities of a diameter, an arc of a great circle, less than a semi-circumference, can be drawn, and but one.*

For the two points, with the centre of the sphere, determine a plane which intersects the surface of the sphere in the required arc.

If the points *are* the extremities of a diameter, an indefinitely great number of arcs of great $\odot$ can be drawn between them; for an indefinitely great number of planes can be drawn through the diameter.

524. Def. The *distance* between two points on the surface of a sphere, not at the extremities of a diameter, is the arc of a great circle, less than a semi-circumference, drawn between them.

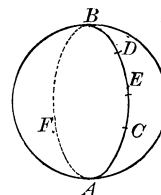
Thus, the distance between points C and D is arc CED , and not arc $CAFBD$.

525. *An arc of a circle may be drawn through any three points on the surface of a sphere.*

For the three points determine a plane which intersects the surface of the sphere in the required arc.

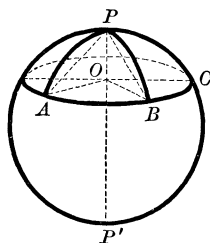
Ex. 1. The radius of a sphere is 18 inches. Find area of circle made by a plane passing through the sphere two inches from the centre.

Ex. 2. Find the locus of points in space three inches from a fixed point.



PROP. II. THEOREM

526. *All points in the circumference of a circle of a sphere are equally distant from each of its poles.*



Given P, P' the poles of $\odot ABC$ of sphere APC ; A, B any two points in circumference ABC , and great $\odot$ arcs PA, PB .

To Prove arc $PA = \text{arc } PB$ (§ 524).

Proof. 1. Let axis PP' intersect plane ABC at O ; draw lines OA, OB , and chords PA, PB .

2. Since O is centre of $\odot ABC$ (§ 519), $OA = OB$. (?)

3. Then chord $PA = \text{chord } PB$ (?), and arc $PA = \text{arc } PB$. (?)

4. In like manner, all points in circumference ABC may be proved equally distant from P' .

527. Def. The *polar distance* of a circle of a sphere is the distance (§ 524) from the nearer of its poles, or from either pole if they are equally near, to the circumference.

Thus, in above figure, polar distance of $\odot ABC$ is arc PA .

528. If, in the figure of Prop. II, ABC is a great $\odot$, and O the centre of the sphere, arc PA is a quadrant. (§ 190)

Hence, *the polar distance of a great circle is a quadrant.*

An arc of a circle may be drawn on the surface of a sphere by placing one foot of the compasses at the nearer pole of the circle, the distance between the feet being equal to the chord of the polar distance.

Ex. 3. Find the locus of points on the surface of a sphere equidistant from the circumferences of two great circles of the sphere.

Ex. 4. If three great circles on a sphere intersect, how many points have their circumferences in common? Is number of points the same if

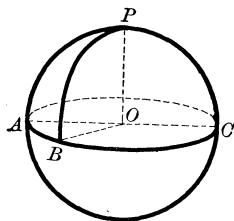
circles are equal small circles? Find greatest possible number of intersection points common to the circumferences of three unequal small circles.

Ex. 5. Mount Rainier is 14,400 feet in height. Considering the earth a sphere whose radius is 4000 miles, at what distance can the peak of this mountain be seen?

PROP. III. THEOREM

529. *If a point on the surface of a sphere lies at a quadrant's distance from each of two points in the arc of a great circle, it is a pole of that arc.*

The term *quadrant*, in Spherical Geometry, usually signifies a quadrant of a great circle.



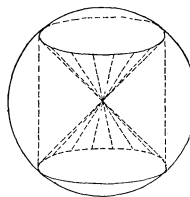
Given point P on surface of sphere APC , AB an arc of great $\odot ABC$, and PA and PB quadrants.

To Prove P a pole of arc AB .

(PO is $\perp$ to OA and OB ; then use § 352.)

Ex. 6. The circumference of a small circle, whose radius is 16, bisects the arcs of great circles drawn from a given great circle to one of its poles. Find the diameter of the sphere.

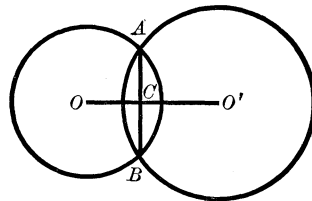
Ex. 7. A cylinder of revolution is inscribed in a sphere whose radius is 18. Lines are drawn from the centre of the sphere to all points in the intersections of the cylinder and the surface of the sphere. If a plane be passed through the axis of the conical surface thus formed, the elements of the two nappes lying in this plane make an angle of 90° . Find volume of solid included between the conical surface of two nappes and the surface of the cylinder.



Ex. 8. In Ex. 7 find the volume of the cylinder, if the radius of the sphere is 12 and the elements make an angle of 60° . Does this statement admit of more than one solution?

PROP. IV. THEOREM

530. *The intersection of two spheres is a circle, whose centre is in the straight line joining the centres of the spheres, and whose plane is perpendicular to that line.*



Given two intersecting spheres.

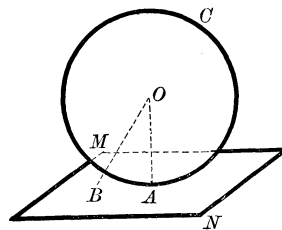
To Prove their intersection a $\odot$, whose centre is in line joining centres of spheres, and whose plane is $\perp$ to this line.

Proof. 1. Let O, O' be centres of two $\odot$; draw line OO' , intersecting common chord AB at C ; OO' bisects AB at rt. $\angle$. (?)

2. If entire figure be revolved about OO' , the $\odot$ generate spheres whose centres are O, O' (§ 516), and AC a $\odot \perp OO'$. (?)

PROP. V. THEOREM

531. *A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*



Given plane $MN \perp$ to radius OA of sphere AC at A .

To Prove MN tangent to sphere.

(Prove as in § 169.)

PROP. VI. THEOREM

532. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.* (Converse of Prop. V.)

Given plane MN tangent to sphere AC at A (Fig. of Prop. V) and radius OA .

To Prove $MN \perp OA$.

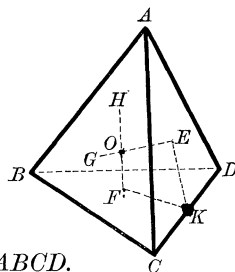
(Prove as in § 170.)

Ex. 9. The radii of two spheres are 9 and 12, respectively. The distances between their centres is 15. Find the area of their circle of intersection.

Ex. 10. If two spheres are tangent to the same plane at the same point, the line joining their centres passes through the point of contact.

PROP. VII. THEOREM

533. *A sphere can be circumscribed about any tetraedron.*



Given tetraedron $ABCD$.

To Prove that a sphere can be circumscribed about it.

Proof. 1. Draw lines KE , KF in faces ACD , BCD , bisecting CD at rt. $\angle$ s, and let E be centre of circumscribed $\odot$ of ACD , and F of BCD . (§ 206)

2. We have plane $EKF \perp CD$ (?), and to faces ACD , BCD . (§ 390)

3. Draw $EG \perp ACD$, $FH \perp BCD$; EG , FH lie in EKF . (§ 388)

4. EG , FH must meet at O , unless $\parallel$, which cannot be unless ACD , BCD are in same plane (§ 368); which is contrary to hyp.

5. O , being in EG , is equally distant from A, C, D ; and being in FH , equally distant from B, C, D . (§ 356, I)

6. Then, a spherical surface with O as centre, and OA as radius, will pass through A, B, C, D .

Ex. 11. Given two planes the plane angle of whose diedral is 60° . Describe a sphere of radius a to which each of these planes shall be tangent. Is there more than one such sphere?

Ex. 12. Through a point at a distance 12 from the centre of a sphere of radius 6 draw a plane tangent to the sphere.

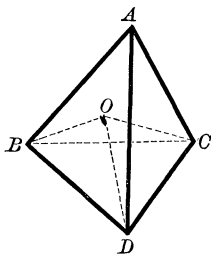
Ex. 13. What is the locus of the centres of spheres which are tangent to each of two parallel planes?

Ex. 14. Three lines not in the same plane intersect at a point. Find the locus of the centres of spheres tangent to the three lines.

Ex. 15. What is the locus of the centres of spheres which have a given radius and whose surfaces pass through a fixed point?

PROP. VIII. THEOREM

534. *A sphere can be inscribed in any tetraedron.*

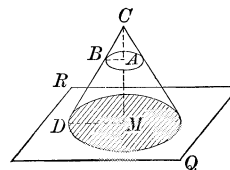


Given tetraedron $ABCD$.

To Prove that a sphere can be inscribed in it.

Proof. If bisecting planes of diedral $\angle BC, BD, CD$ meet at point O , this point is equally distant from faces of tetraedron. (§ 392)

Ex. 16. A light C is equally distant from all points in the circumference of an opaque circle whose radius is AB . The ray of light, line CBD , makes an angle of 60° with the plane QR , parallel to circle A . AB is 8, and the area of shadow on QR is 576π . Find volume of frustum of cone whose bases are circle A and the shadow.



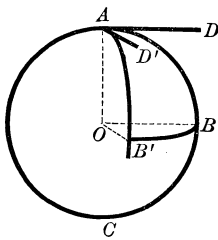
Ex. 17. A system of spheres is tangent to the faces of a trihedral. Each face angle of the trihedral is 90° . Find locus of the centres of the spheres.

535. Defs. The angle formed by two intersecting curves is the angle formed by tangents to the curves at their point of intersection.

A *spherical angle* is the angle formed by two intersecting arcs of great circles.

PROP. IX. THEOREM

536. A spherical angle is measured by an arc of a great circle having its vertex as a pole, included between its sides produced if necessary.



Given AB, AB' arcs of great $\odot$ on sphere AC ; AD, AD' tangent to AB, AB' , respectively, and BB' an arc of great $\odot$ with A as pole, included between AB, AB' .

To Prove $\angle DAD'$ measured by arc BB' .

Proof. 1. Draw radii OA, OB, OB' ; since arcs AB, AB' are quadrants (§ 528), $\angle AOB, \angle AOB'$ are rt. $\angle$ s. (?)

2. Then $OB \parallel AD, OB' \parallel AD'$ (?); and $\angle DAD' = \angle BOB'$. (§ 376)

3. $\angle BOB'$ is measured by arc BB' (?); then $\angle DAD'$ is.

537. Since plane BOB' (Fig. of Prop. IX) is $\perp$ OA (§ 352), planes AOB and BOB' are $\perp$. (§ 390)

Now a tangent to arc AB at B is $\perp$ BOB' . (§ 388)

Then it is $\perp$ to a tangent to arc BB' at B . (§ 350)

Then, spherical $\angle ABB'$ is a rt. $\angle$. (§ 535)

That is, *an arc of a great circle drawn from the pole of a great circle is perpendicular to its circumference.*

538. By § 536, *the angle formed by two arcs of great circles is the plane angle of the dihedral angle between their planes.* (§ 379)

SPHERICAL POLYGONS AND SPHERICAL PYRAMIDS

DEFINITIONS

539. A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of *great circles*; as $ABCD$.

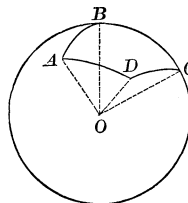
The bounding arcs are called the *sides* of the spherical polygon, and are usually measured in *degrees*.

The *angles* of the spherical polygon are the spherical angles (§ 535) formed by the adjacent sides, and their vertices are called the *vertices* of the spherical polygon.

A *diagonal* of a spherical polygon is an arc of a great circle joining any two vertices which are not consecutive.

A *spherical triangle* is a spherical polygon of three sides.

A spherical triangle is called *isosceles* when it has two sides equal; *equilateral* when all its sides are equal; and *right* when it has a right angle.



540. The planes of the sides of a spherical polygon form a polyedral angle, whose vertex is the centre of the sphere, and whose face angles are measured by the sides of the spherical polygon (§ 191).

Thus, in figure § 539, the planes of the sides of the spherical polygon form a polyedral angle, $O-ABCD$, whose face $\angle AOB$, BOC , etc., are measured by arcs AB , BC , etc., respectively.

A spherical polygon is called *convex* when the polyedral angle formed by the planes of its sides is convex (§ 402).

541. A *spherical pyramid* is a solid bounded by a spherical polygon and the planes of its sides; as $O-ABCD$, figure § 539.

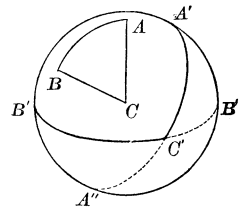
The centre of the sphere is called the *vertex* of the spherical pyramid, and the spherical polygon the *base*.

542. Two spherical pyramids are equal when their bases are equal.

For they can be applied one to the other so as to coincide throughout.

543. If circumferences of great circles be drawn with the vertices of a spherical triangle as poles, they divide the surface of the sphere into eight spherical triangles.

Thus, if circumference $B'C'B''$ be drawn with vertex A of spherical $\triangle ABC$ as pole, circumference $A'C'A''$ with B as pole, and circumference $A'B'A''B'$ with C as pole, the surface of the sphere is divided into eight spherical $\triangle$; $A'B'C'$, $A'B''C'$, $A''B'C'$, and $A''B''C'$ on the hemisphere represented in the figure, the others on the opposite hemisphere.



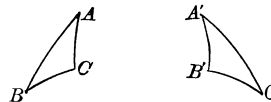
Of these eight spherical $\triangle$, one is called the *polar triangle* of ABC , and is determined as follows:

Of the intersections, A' and A'' , of circumferences drawn with B and C as poles, that which is less than a quadrant's distance from A , *i.e.* A' , is a vertex of the polar triangle; and similarly for the other intersections.

Thus, $A'B'C'$ is the polar $\triangle$ of ABC .

544. Two spherical polygons, on the same or equal spheres, are said to be *symmetrical* when the sides and angles of one are equal, respectively, to the sides and angles of the other, if the equal parts occur *in the reverse order*.

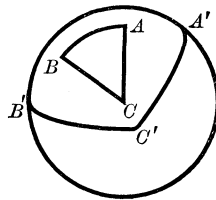
Thus, if spherical $\triangle ABC$ and $\triangle A'B'C'$, on the same or equal spheres, have sides AB, BC, CA equal, respectively, to sides $A'B', B'C', C'A'$, and $\angle A, B, C$ to $\angle A', B', C'$, and the equal parts occur in the reverse order, the $\triangle$ are symmetrical.



It is evident that, in general, two symmetrical spherical polygons cannot be placed so as to coincide throughout.

PROP. X. THEOREM

545. *If one spherical triangle is the polar triangle of another, the second spherical triangle is the polar triangle of the first.*



Given $\triangle A'B'C'$ the polar $\triangle$ of spherical $\triangle ABC$; A, B, C being the poles of arcs $B'C', C'A', A'B'$, respectively.

To Prove $\triangle ABC$ the polar $\triangle$ of spherical $\triangle A'B'C'$.

Proof. 1. Since B is the pole of $A'C'$, and C of $A'B'$, A' is a quadrant's distance from B and C . (§ 528)

2. Then A' is pole of arc BC . (§ 529)

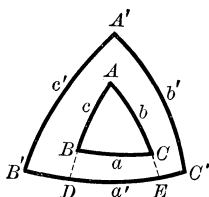
3. Similarly, B' is pole of CA , and C' of AB .

4. Then, $\triangle ABC$ is polar $\triangle$ of $\triangle A'B'C'$; for of the two intersections of circumferences with B', C' as poles, A is $<$ a quadrant from A' ; and similarly for other vertices. (§ 543)

Two spherical triangles, each of which is the polar triangle of the other, are called *polar triangles*.

PROP. XI. THEOREM

546. *In two polar triangles, each angle of one is measured by the supplement of that side of the other of which it is the pole.*



Given $A, B, C, A', B',$ and C' the $\angle$ s, in degrees, of polar $\triangle ABC, A'B'C'$.

Let A be the pole of $B'C'$, B of $C'A'$, C of $A'B'$, A' of BC , B' of CA , and C' of AB .

Let BC expressed in degrees be denoted by a , CA by b , AB by c , $B'C'$ by a' , $C'A'$ by b' , $A'B'$ by c' .

To Prove

$$A = 180^\circ - a', \quad B = 180^\circ - b', \quad C = 180^\circ - c',$$

$$A' = 180^\circ - a, \quad B' = 180^\circ - b, \quad C' = 180^\circ - c.$$

Proof. 1. Extend arcs AB, AC to meet $B'C'$ at D and E .

2. Arcs $B'E, C'D$ are quadrants. (§ 528)

3. Then $B'E + C'D = 180^\circ$, or $DE + B'C' = 180^\circ$.

4. Since A is pole of $B'C'$, arc DE measures $\angle A$. (§ 536)

5. Then result (3) becomes $A + a' = 180^\circ$, or $A = 180^\circ - a'$.

6. Similarly we prove theorem for any $\angle$ of either $\triangle$.

547. Def. Two spherical polygons on the same or equal spheres are said to be *mutually equilateral*, or *mutually equiangular*, when the sides or angles of one are equal, respectively, to the homologous sides or angles of the other, *whether taken in the same or in the reverse order*.

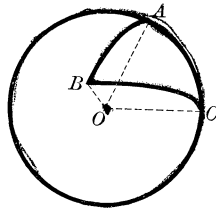
548. It follows from § 546 that *if two spherical triangles on*

the same or equal spheres are mutually equiangular, their polar triangles are mutually equilateral.

For any two homologous sides of the polar $\triangle$ are supplements of equal $\angle$ s, and therefore equal.

PROP. XII. THEOREM

549. *Any side of a spherical triangle is less than the sum of the other two sides.*



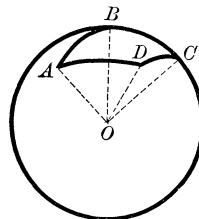
Given AB any side of spherical $\triangle ABC$.

To Prove $AB < AC + BC$.

(By § 406, $\angle AOB < \angle AOC + \angle BOC$; these $\angle$ s are measured by AB , AC , BC , respectively.)

PROP. XIII. THEOREM

550. *The sum of the sides of a convex spherical polygon is less than 360° .*



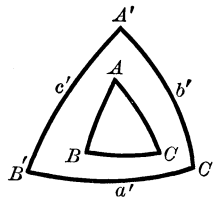
Given convex spherical polygon $ABCD$.

To Prove $AB + BC + CD + DA < 360^\circ$.

(By § 407, sum of $\angle$ s AOB , BOC , COD , and DOA is $< 360^\circ$.)

PROP. XIV. THEOREM

551. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*



Given spherical $\triangle ABC$.

To Prove $A + B + C > 180^\circ$, and $< 540^\circ$.

Proof. 1. Let $A'B'C'$ be polar $\triangle$ of ABC ; a', b', c' being sides having A, B, C , respectively, as poles.

$$\begin{aligned} 2. \text{ By } \S 546, \quad A &= 180^\circ - a', \\ B &= 180^\circ - b', \\ C &= 180^\circ - c'. \end{aligned}$$

$$\begin{aligned} 3. \text{ Adding, } A + B + C &= 540^\circ - (a' + b' + c'); \quad (1) \\ \text{therefore, } A + B + C &< 540^\circ. \end{aligned}$$

$$\begin{aligned} -4. \text{ Since } a' + b' + c' \text{ is } < 360^\circ \text{ (} \S 550 \text{), by (1),} \\ A + B + C &> 180^\circ. \end{aligned}$$

552. By $\S 551$, a spherical $\triangle$ may have one, two, or three rt. $\angle$ s, or one, two, or three obtuse $\angle$ s.

553. Def. We call a spherical triangle having two right angles a *bi-rectangular triangle*, and one having three right angles a *tri-rectangular triangle*.

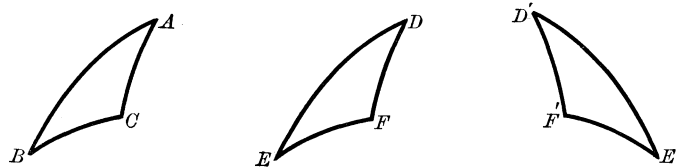
Ex. 18 Each side of a spherical triangle is 90° . Find angles and sides of its polar triangle.

Ex. 19. The angles of a spherical triangle are 93° , 128° , and 117° , respectively. Find number of degrees in each side of its polar triangle.

PROP. XV. THEOREM

554. *If two spherical triangles on the same or equal spheres have two sides and the included angle of one equal, respectively, to two sides and the included angle of the other,*

- I. *They are equal if the equal parts occur in the same order.*
- II. *They are symmetrical if the equal parts occur in the reverse order.*



I. **Given** ABC , DEF spherical $\triangle$ on the same or equal spheres, having

$$AB = DE, AC = DF, \text{ and } \angle A = \angle D;$$

the equal parts occurring in the same order.

To Prove $\triangle ABC = \triangle DEF$.

Proof. Superpose $\triangle ABC$ upon $\triangle DEF$ so that AB shall fall on DE , and AC on DF ; BC falls on EF by § 523.

II. **Given** ABC , $D'E'F'$ spherical $\triangle$ on the same or equal spheres, having

$$AB = D'E', AC = D'F', \text{ and } \angle A = \angle D';$$

the equal parts occurring in the reverse order.

To Prove ABC and $D'E'F'$ symmetrical.

Proof. 1. Make $\triangle DEF$, on the same or an equal sphere symmetrical to $D'E'F'$, having

$$DE = D'E', DF = D'F', \text{ and } \angle D = \angle D';$$

the equal parts occurring in the reverse order.

2. Then $AB = DE$, $AC = DF$, and $\angle A = \angle D$ (?); whence $\triangle ABC = \triangle DEF$ (§ 554, I) and is symmetrical to $D'E'F'$.

PROP. XVI. THEOREM

555. *If two spherical triangles on the same or equal spheres have a side and two adjacent angles of one equal, respectively, to a side and two adjacent angles of the other,*

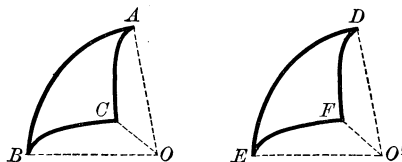
- I. *They are equal if the equal parts occur in the same order.*
- II. *They are symmetrical if the equal parts occur in the reverse order.*

(The proof is left to the pupil; use Fig. of Prop. XV.)

PROP. XVII. THEOREM

556. *If two spherical triangles on the same or equal spheres are mutually equilateral,*

- I. *They are equal if the equal parts occur in the same order.*
- II. *They are symmetrical if the equal parts occur in the reverse order.*



Given ABC , DEF mutually equilateral spherical $\triangle$ on equal spheres with side BC equal to EF .

To Prove ABC and DEF equal or symmetrical according as the equal parts occur in the same or in the reverse order.

Proof. 1. Draw radii of spheres OA , OB , OC , $O'D$, $O'E$, $O'F$.

2. Since triedrals $O-ABC$, $O'-DEF$ have homologous face $\angle$ equal (?), diedrals OA , $O'D$ are equal. (§ 408)

3. Then plane $\angle$ of OA , $O'D$ are equal. (§ 385)

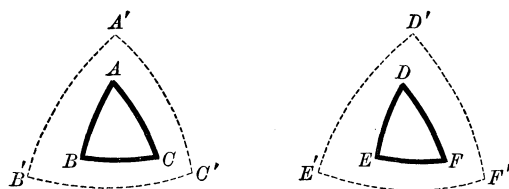
4. Then, $\angle BAC = \angle EDF$ (§ 538); and theorem is proved by § 554.

The theorem may be proved in a similar manner when given spherical $\triangle$ are on same sphere.

PROP. XVIII. THEOREM

557. *If two spherical triangles on the same or equal spheres are mutually equiangular,*

- I. *They are equal if the equal parts occur in the same order.*
- II. *They are symmetrical if the equal parts occur in the reverse order.*



Given ABC , DEF mutually equiangular spherical $\triangle$ on the same or equal spheres.

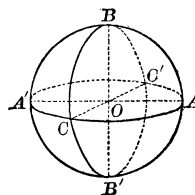
To Prove ABC and DEF equal or symmetrical according as the equal parts occur in the same or in the reverse order.

Proof. 1. Let $A'B'C'$, $D'E'F'$ be polar $\triangle$ of ABC , DEF .

2. $\triangle A'B'C'$, $D'E'F'$ are mutually equilateral (§ 548); then, homologous $\angle$ s of $A'B'C'$, $D'E'F'$ are equal (§ 556).

3. Since ABC is polar $\triangle$ of $A'B'C'$, and DEF of $D'E'F'$ (?), ABC , DEF are mutually equilateral (§ 548); and theorem follows by § 556.

558. *If three diameters of a sphere be so drawn that each is perpendicular to the other two, the planes determined by them divide the surface of the sphere into eight equal tri-rectangular triangles.*



For by § 537, each $\angle$ of each spherical $\triangle$ is a rt. $\angle$, and the $\triangle$ are equal by § 557.

559. By § 558, *the surface of a sphere is eight times the surface of one of its tri-rectangular triangles.*

PROP. XIX. THEOREM

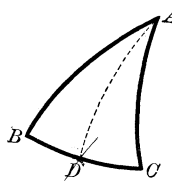
560. *In an isosceles spherical triangle the angles opposite the equal sides are equal.*

Given, in spherical $\triangle ABC$, $AB = AC$.

To Prove $\angle B = \angle C$.

Proof. 1. Draw AD an arc of a great $\odot$, bisecting BC at D .

2. By § 556, II, $\triangle ABD$, $\triangle ACD$ are symmetrical; and $\angle B = \angle C$.



561. *An isosceles spherical triangle is equal to the spherical triangle which is symmetrical to it.*

For the equal parts occur in the same order.

PROP. XX. THEOREM

562. *If two angles of a spherical triangle are equal, the sides opposite are equal.* (Converse of Prop. XIX.)

Given in spherical $\triangle ABC$, $\angle B = \angle C$.

To Prove $AB = AC$.

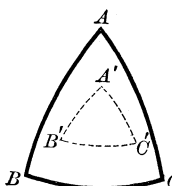
Proof. 1. Let $A'B'C'$ be polar $\triangle$ of ABC ; B being pole of $A'C'$, and C of $A'B'$.

2. Since $A'C'$ is sup. of $\angle B$, and $A'B'$ of $\angle C$ (?),

$$A'C' = A'B' \text{ (?), and } \angle B' = \angle C'. \quad (\S 560)$$

3. Since AB is sup. of $\angle C'$, and AC of $\angle B'$ (§ 545),

$$AB = AC.$$



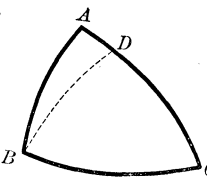
PROP. XXI. THEOREM

563. *If two angles of a spherical triangle are unequal, the sides opposite are unequal, and the greater side lies opposite the greater angle.*

Given, in spherical $\triangle ABC$, $\angle ABC > \angle C$.

To Prove $AC > AB$.

(Compare § 95. Draw BD an arc of a great $\odot$ meeting AC at D , and making $\angle CBD$ equal to $\angle C$.)



PROP. XXII. THEOREM

564. *If two sides of a spherical triangle are unequal, the angles opposite are unequal, and the greater angle lies opposite the greater side. (Converse of Prop. XXI.)*

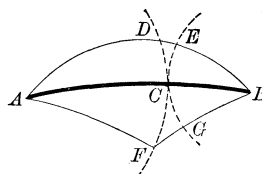
(Prove by *Reductio ad absurdum*.)

PROP. XXIII. THEOREM

565. *The shortest line on the surface of a sphere between two given points is the arc of a great circle, not greater than a semi-circumference, which joins the points.*

Given points A and B on the surface of a sphere, and AB an arc of a great $\odot$, not $>$ a semi-circumference.

To Prove AB the shortest line on the surface of the sphere between A and B .



Proof. 1. Let C be any point on arc AB ; and DCF , ECG arcs of small $\odot$ with A , B as poles, and AC , BC as polar distances.

2. Arcs DCF , ECG have only point C common; for if F is any other point in DCF , and AF , BF arcs of great $\odot$,

$$AF = AC. \quad (\S 526)$$

3. By § 549, $AF + BF > AC + BC$;
subtracting AF from first member, and AC from second, we have $BF > BC$ (or BG), and F is without $\odot ECG$.

4. To prove shortest line on surface passes through C , let any other line on surface, $ADEB$, not passing through C , cut arc DCF at D , and arc ECG at E .

5. Whatever the nature of AD , an equal line can be drawn from A to C ; and a line equal to BE from B to C .

6. Then a line can be drawn from A to B through C , equal to $AD + BE$, and hence $< ADEB$ by line DE .

7. Then no line which does not pass through C can be the shortest line from A to B ; and C being *any* point in AB , the shortest line passes through *every* point of AB .

Ex. 20. A right cone is divided into two equivalent parts by a plane which is drawn parallel to its base. In what ratio is the altitude of the cone divided by the plane?

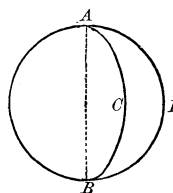
MEASUREMENT OF SPHERICAL POLYGONS

566. Defs. A *lune* is a portion of the surface of a sphere bounded by two semi-circumferences of great circles; as $ACBD$.

The *angle* of the lune is the angle formed by its bounding arcs.

A *spherical wedge* is a solid bounded by a lune and the planes of its bounding arcs.

The lune is called the *base* of the spherical wedge.

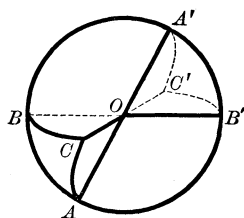


567. It is evident that *two lunes on the same sphere, or equal spheres, are equal when their angles are equal.*

Also, *two spherical wedges in the same sphere, or equal spheres, are equal when the angles of the lunes which form their bases are equal.*

PROP. XXIV. THEOREM

568. *The spherical triangles corresponding to a pair of vertical triedral angles are symmetrical.*



Given AOA' , BOB' , COC' diameters of sphere AC ; also, planes determined by them, intersecting the surface in arcs AB , BC , CA , $A'B'$, $B'C'$, $C'A'$.

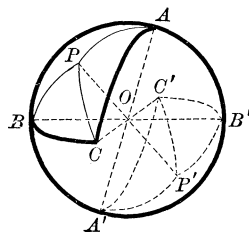
To Prove spherical $\triangle ABC$, $A'B'C'$ symmetrical.

Proof. 1. Since $\angle AOB = \angle A'OB'$ (?), $AB = A'B'$. (?)

2. Similarly, $BC = B'C'$, $CA = C'A'$; now use § 556, 2.

PROP. XXV. THEOREM

569. *Two spherical triangles corresponding to a pair of vertical triedral angles are equivalent.*



Given AOA' , BOB' , COC' diameters of sphere AB ; also, planes determined by them, intersecting the surface in arcs AB , BC , CA , $A'B'$, $B'C'$, $C'A'$.

To Prove $\text{area } ABC = \text{area } A'B'C'$.

Proof. 1. Let P be pole of small $\odot$ passing through A , B , C ; draw diameter of sphere PP' , and arcs of great $\odot$ PA , PB , PC , $P'A'$, $P'B'$, $P'C'$; then $PA = PB = PC$. (§ 526)

2. $\triangle PAB$, $P'A'B'$ are symmetrical (§ 568); whence

$$\triangle PAB = \triangle P'A'B'. \quad (\S 561)$$

3. Similarly, $\triangle PBC = \triangle P'B'C'$, $\triangle PCA = \triangle P'C'A'$.

4. $\triangle PAB$, PBC , PCA compose $\triangle ABC$, and $\triangle P'A'B'$, $P'B'C'$, $P'C'A'$ compose $\triangle A'B'C'$; which proves theorem.

If P and P' fall without spherical $\triangle ABC$ and $A'B'C'$, we should take the sum of two isosceles spherical $\triangle$, diminished by the third.

570. *Two symmetrical spherical triangles are equivalent.*

571. Since spherical pyramids $O-APB$, $O-BPC$, $O-CPA$ equal, respectively, spherical pyramids $O-A'P'B'$, $O-B'P'C'$, $O-C'P'A'$ (§ 542), vol. $O-ABC = \text{vol. } O-A'B'C'$.

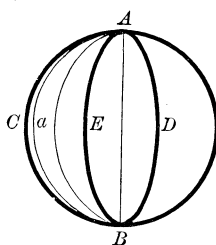
Whence, the spherical pyramids corresponding to a pair of vertical triedral angles are equivalent.

PROP. XXVI. THEOREM

572. *Two lunes on the same sphere, or equal spheres, are to each other as their angles.*

The word “*lune*,” in the above statement, signifies the *area* of the lune.

Case I. *When the angles are commensurable.*



Given $ACBD$, $ACBE$ lunes on sphere AB , having their $\angle CAD$, $\angle CAE$ commensurable.

To Prove $\frac{ACBD}{ACBE} = \frac{\angle CAD}{\angle CAE}$. (1)

Proof. 1. Let $\angle CAa$ be contained 5 times in $\angle CAD$ and 3 times in $\angle CAE$; find ratio $\angle CAD$ to $\angle CAE$.

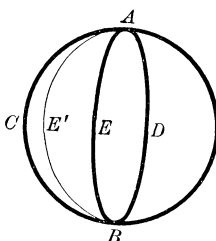
2. Extend arcs of division of $\angle CAD$ to B ; subdivisions of $ACBD$, $ACBE$ are all equal. (§ 567)

3. Find ratio $ACBD$ to $ACBE$.

4. From results (1) and (3), we have equation (1).

The theorem may be proved in a similar manner when the given lunes are on equal spheres.

Case II. *When the angles are incommensurable.*



Given $ACBD$, $ACBE$ lunes on sphere AB , having $\angle CAD$, $\angle CAE$ incommensurable.

To Prove
$$\frac{ACBD}{ACBE} = \frac{\angle CAD}{\angle CAE}.$$

(Prove as in §§ 188 or 227. Let $\angle CAD$ be divided into any number of equal parts, and apply one of these parts to $\angle CAE$ as a unit of measure.)

573. *The surface of a lune is to the surface of the sphere as the angle of the lune is to four right angles.*

For the surface of a sphere may be regarded as a lune whose $\angle$ is equal to 4 rt. $\angle$ s.

574. *If the unit of measure for angles is the right angle, the area of a lune is equal to twice its angle, multiplied by the area of a tri-rectangular triangle.*

Let L be area of lune, A the numerical measure of its $\angle$ referred to a rt. $\angle$ as unit of measure, T the area of a tri-rectangular triangle; since area surface of sphere is 8 T (§ 559),

$$\frac{L}{8T} = \frac{A}{4} \text{ (§ 572); and } L = \frac{A}{4} \times 8T = 2A \times T.$$

Let it be required, for example, to find the area of a lune whose $\angle$ is 50° , on a sphere the area of whose surface is 72.

The $\angle$ of the lune referred to a rt. $\angle$ as the unit of measure is $\frac{5}{9}$; and T is $\frac{1}{8}$ of 72, or 9. Then the area of the lune is $2 \times \frac{5}{9} \times 9$, or 10.

575. Def. A *tri-rectangular pyramid* is a spherical pyramid whose base is a tri-rectangular triangle.

576. It may be proved, as in § 572, that

Two spherical wedges in the same sphere, or equal spheres, are to each other as the angles of the lunes which form their bases.

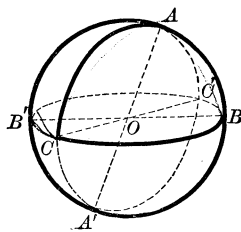
577. It may be proved that *if the unit of measure for angles is the right angle, the volume of a spherical wedge is equal to twice the angle of the lune which forms its base, multiplied by the volume of a tri-rectangular pyramid.*

578. Def. The *spherical excess* of a spherical triangle is the excess of the sum of its angles above 180° (§ 551).

Thus, if the $\angle$ s of a spherical $\triangle$ are 65° , 80° , and 95° , its spherical excess is $65^\circ + 80^\circ + 95^\circ - 180^\circ$, or 60° .

PROP. XXVII. THEOREM

579. *If the unit of measure for angles is the right angle, the area of a spherical triangle is equal to its spherical excess, multiplied by the area of a tri-rectangular triangle.*



Given A, B, C the numerical measures of the $\angle$ s of spherical $\triangle ABC$, referred to a rt. $\angle$ as the unit, and T the area of a tri-rectangular $\triangle$.

To Prove $\text{area } ABC = (A + B + C - 2) \times T.$ (1)

Proof. 1. Draw diameters AA', BB', CC' , and circumferences $ABA'B', ACA'C', BCB'C'$; $ABA'B'$ is a lune whose $\angle$ is A .

2. By § 574, $\text{area } ABC + \text{area } A'BC = 2 A \times T.$ (2)

3. Similarly, $\text{area } ABC + \text{area } AB'C = 2 B \times T.$ (3)

4. Lune $ACBC'$, whose $\angle$ is C , is composed of $\triangle ABC$, ABC' ; and $\text{area } ABC' = \text{area } A'B'C.$ (§ 569)

5. Then, $\text{area } ABC + \text{area } A'B'C = 2 C \times T.$ (4)

6. Add (2), (3), (4); since $\triangle ABC, A'BC, AB'C, A'B'C$ form surface of hemisphere whose area is $4 T$ (§ 559),

$$2 \text{ area } ABC + 4 T = (2 A + 2 B + 2 C) \times T.$$

7. Transposing $4 T$, and dividing by 2, we obtain equation (1).

Let it be required, for example, to find the area of a spherical $\triangle$ whose $\angle$ s are 105° , 80° , and 95° , on a sphere the area of whose surface is 144.

The spherical excess of the spherical $\triangle$ is 100° , or $\frac{10}{9}$ referred to a rt. $\angle$ as the unit of measure; and the area of a tri-rectangular $\triangle$ is $\frac{1}{8}$ of 144, or 18.

Then the area of the spherical $\triangle$ is $\frac{10}{9} \times 18$, or 20.

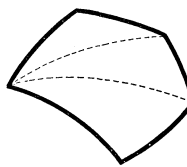
580. It may be proved, as in § 579, that

If the unit of measure for angles is the right angle, the volume of a triangular spherical pyramid is equal to the spherical excess of its base, multiplied by the volume of a tri-rectangular pyramid.

PROP. XXVIII. THEOREM

581. *If the unit of measure for angles is the right angle, the area of any spherical polygon is equal to the sum of its angles, diminished by as many times two right angles as the figure has sides less two, multiplied by the area of a tri-rectangular triangle.*

Given K the area of any spherical polygon, n the number of its sides, s the sum of its $\angle$ s referred to a rt. $\angle$ as the unit, and T the area of a tri-rectangular Δ .



To Prove $K = [s - 2(n - 2)] \times T$. (1)

Proof. 1. Divide spherical polygon into $n - 2$ spherical Δ by drawing diagonals from any vertex.

2. By § 579, area each Δ equals sum of its $\angle$ s, minus 2 rt. $\angle$ s, multiplied by T .

3. Then, sum of areas of Δ equals sum of their $\angle$ s (or sum of $\angle$ s of polygon), minus $n - 2$ times 2 rt. $\angle$ s, multiplied by T ; which proves equation (1).

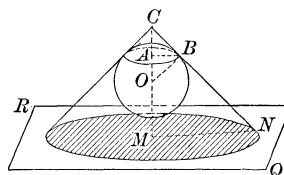
582. It may be proved, as in § 581, that

If the unit of measure for angles is the right angle, the volume of any spherical pyramid is equal to the sum of the angles of its base, diminished by as many times two right angles as the base has sides less two, multiplied by the volume of a tri-rectangular pyramid.

Ex. 21. The sides of a spherical triangle, on a sphere the area of whose surface is 156, are 44° , 63° , and 97° , respectively. Find the area of its polar triangle.

Ex. 22. Find the area of a spherical hexagon whose angles are 120° , 139° , 148° , 155° , 162° , and 167° , on a sphere the area of whose surface is 280.

Ex. 23. Given an opaque sphere whose radius OB is $16\sqrt{2}$ feet. A light is placed at C , a point so chosen that $\angle AOB = 45^\circ$, AB being the radius of the small circle which intercepts the light. The shadow cast by the sphere upon the plane QR is a circle with centre at M , and has an area of 9216π square feet. Find the distance, CM , from the light to the plane QR .



If M were *not* the centre of the shadow and the shadow were *not* a circle, would the circle whose centre is A and the plane QR be parallel?

Ex. 24. Find the volume of a pentagonal spherical pyramid the angles of whose base are 109° , 128° , 137° , 163° , and 158° , respectively; the volume of the sphere being 180.

MEASUREMENT OF THE SPHERE

DEFINITIONS

583. A *zone* is a portion of the surface of a sphere included between two parallel planes.

The circumferences of the circles which bound the zone are called the *bases*, and the perpendicular distance between their planes the *altitude*.

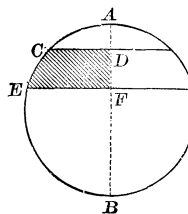
A *zone of one base* is a zone one of whose bounding planes is tangent to the sphere.

A *spherical segment* is a portion of a sphere included between two parallel planes.

The circles which bound it are called the *bases*, and the perpendicular distance between them the *altitude*.

A *spherical segment of one base* is a spherical segment one of whose bounding planes is tangent to the sphere.

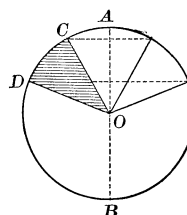
584. If semicircle $ACEB$ be revolved about diameter AB as an axis, and CD and EF are lines $\perp AB$, arc CE generates a zone whose altitude is DF , $CEFD$ a spherical segment whose altitude is DF , arc AC a zone of one base, and ACD a spherical segment of one base.



585. If a semicircle be revolved about its diameter as an axis, the solid generated by any sector of the semicircle is called a *spherical sector*.

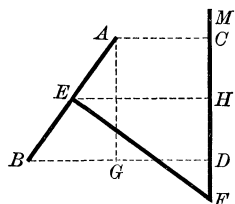
Thus, if semicircle $ACDB$ be revolved about diameter AB as an axis, sector OCD generates a spherical sector.

The zone generated by the arc of the sector is called the *base* of the spherical sector.



PROP. XXIX. THEOREM

586. The area of the surface generated by the revolution of a straight line about a straight line in its plane, not parallel to and not intersecting it, as an axis, is equal to its projection on the axis, multiplied by the circumference of a circle, whose radius is the perpendicular erected at the middle point of the line and terminating in the axis.



Given str. line AB revolved about str. line FM in its plane, not $\parallel$ to and not intersecting it, as an axis; lines AC and $BD \perp FM$, and EF the $\perp$ erected at the middle point of AB terminating in FM .

To Prove area $AB^* = CD \times 2\pi EF$ (§§ 254, 334). (1)

Proof. 1. Draw line $AG \perp BD$, line $EH \perp CD$; AB generates lateral surface of a frustum of a cone of revolution.

2. By § 509, area $AB = AB \times 2\pi EH$. (2)

3. Since $\triangle ABG, EFH$ are similar (?), $\frac{AB}{AG} = \frac{EF}{EH}$. (?)

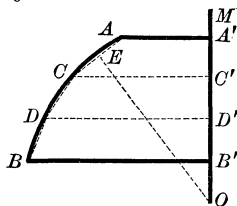
4. Then, $AB \times EH = AG \times EF = CD \times EF$. (?)

5. Substituting in (2), gives equation (1).

*The expression "area AB " is used to denote the area of the surface generated by AB .

PROP. XXX. THEOREM

587. *The area of a zone is equal to its altitude multiplied by the circumference of a great circle.*



Given arc AB revolved about diameter OM as an axis, lines $AA', BB' \perp OM$, and R the radius of the arc.

To Prove area zone generated by $AB = A'B' \times 2\pi R$. (1)

Proof. 1. Divide AB into equal arcs AC, CD, DB ; draw chords AC, CD, DB , lines $CC', DD' \perp OM$, and $OE \perp AC$.

2. By § 586, $\text{area } AC = A'C' \times 2\pi OE$,
 $\text{area } CD = C'D' \times 2\pi OE$, etc.

3. Adding, area generated by broken line $ACDB$
 $= (A'C' + C'D' + \text{etc.}) \times 2\pi OE = A'B' \times 2\pi OE$.

4. If subdivisions of AB be bisected indefinitely, area generated by broken line $ACDB$ approaches area generated by arc AB as a limit,* and $A'B' \times 2\pi OE$ approaches $A'B' \times 2\pi R$ as a limit. (§ 330†)

5. Prove equation (1) by Theorem of Limits.

588. The proof of § 587 holds for any zone which lies entirely on the surface of a hemisphere; for, in that case, no chord is $\parallel OM$, and § 586 is applicable.

Since a zone not entirely on the surface of a hemisphere is the sum of two zones, each of which does lie entirely on the surface of a hemisphere, the theorem of § 587 holds for any zone.

* A rigorous proof of this is beyond the scope of the present treatise.

† The theorem of § 330 is evidently true when, instead of the perimeter of a regular inscribed polygon, we have a broken line consisting of equal straight lines inscribed in an arc.

589. If S denotes the area of a zone, h its altitude, R the radius of the sphere,

$$S = 2 \pi R h.$$

590. Since the surface of a sphere may be regarded as a zone whose altitude is a diameter of the sphere,

The area of the surface of a sphere is equal to its diameter multiplied by the circumference of a great circle.

591. Let S denote the area of the surface of a sphere, R its radius, and D its diameter.

Then,
$$S = 2 R \times 2 \pi R = 4 \pi R^2.$$

That is, *the area of the surface of a sphere is equal to the square of its radius multiplied by 4π .*

Again,
$$S = \pi \times (2 R)^2 = \pi D^2.$$

That is, *the area of the surface of a sphere is equal to the square of its diameter multiplied by π .*

592. Since πR^2 is the area of a great $\odot$ (?), *the surface of a sphere is equivalent to four great circles.*

593. *The areas of the surfaces of two spheres are to each other as the squares of their radii, or as the squares of their diameters.*

(The proof is left to the pupil; compare § 338.)

Ex. 25. Find the area of the surface of the earth included in the north temperate zone. (Call earth a sphere whose radius is 4000 miles and the altitude of the zone 2164.8 miles.)

Ex. 26. A sphere is 32 feet in diameter. Find the area of the surface visible to an observer standing 32 feet from the centre of the sphere.

Ex. 27. The radii of two spheres are in the ratio $\frac{3}{4}$. What is the ratio of the areas of their surfaces?

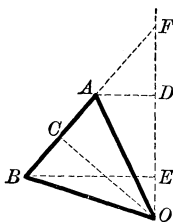
Ex. 28. The area of the surface of a sphere is three times the area of the surface of another. Find the ratio of their radii. If the area of one is M times that of the other, find ratio of radii.

Ex. 29. Find the area of a spherical triangle whose angles are 75° , 105° , and 100° , respectively, on a sphere whose radius is 8 inches.

Ex. 30. A plane bisects a radius of a sphere. Find the ratio of the areas of the surfaces into which the surface of the sphere is divided.

PROP. XXXI. THEOREM

594. *If an isosceles triangle be revolved about a straight line in its plane, not parallel to its base, as an axis, which passes through its vertex without intersecting its surface, the volume of the solid generated is equal to the area of the surface generated by the base, multiplied by one-third the altitude.*



Given isosceles $\triangle OAB$ revolved about str. line OF in its plane, not $\parallel$ to base AB , as an axis; and line $OC \perp AB$.

To Prove $\text{vol. } OAB^* = \text{area } AB \times \frac{1}{3} OC.$ (1)

Proof. 1. Draw $AD, BE \perp OF$; extend BA to meet OF at F .

2. $\text{vol. } OBF = \text{vol. } OBE + \text{vol. } BEF$

$$= \frac{1}{3} \pi \overline{BE}^2 \times OE + \frac{1}{3} \pi \overline{BE}^2 \times EF \quad (\S 505)$$

$$= \frac{1}{3} \pi \overline{BE}^2 \times (OE + EF) = \frac{1}{3} \pi BE \times BE \times OF.$$

3. Since $BE \times OF = OC \times BF$, each being twice area OBF ,
 $\text{vol. } OBF = \frac{1}{3} \pi BE \times OC \times BF = \pi BE \times BF \times \frac{1}{3} OC.$

4. Then (§ 503), $\text{vol. } OBF = \text{area generated by } BF \times \frac{1}{3} OC.$ (2)

5. Similarly, $\text{vol. } OAF = \text{area generated by } AF \times \frac{1}{3} OC.$ (3)

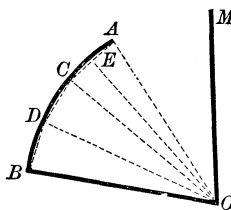
6. Subtracting (3) from (2) gives equation (1).

Ex. 31. The area of the surface of a sphere is 900π . Find volume of an inscribed cone whose axis coincides with a fixed diameter and the radius of whose base is 22. How many cones satisfy the condition?

* The expression “ $\text{vol. } OAB$ ” is used to denote *the volume of the solid generated by* OAB .

PROP. XXXII. THEOREM

595. *The volume of a spherical sector is equal to the area of the zone which forms its base, multiplied by one-third the radius of the sphere.*



Given sector OAB revolved about diameter OM as an axis, and R the radius of the arc.

To Prove volume of spherical sector generated by OAB equal to area of zone generated by $AB \times \frac{1}{3} R$.

Proof. 1. Divide AB into equal arcs, AC , CD , DB ; draw chords AC , CD , DB , also lines OC , OD , and line $OE \perp AC$.

2. By § 594, vol. $OAC = \text{area } AC \times \frac{1}{3} OE$,
 vol. $OCD = \text{area } CD \times \frac{1}{3} OE$, etc.

3. Adding, volume generated by polygon $OACDB$
 $= (\text{area } AC + \text{area } CD, \text{ etc.}) \times \frac{1}{3} OE = \text{area } ACDB \times \frac{1}{3} OE$.

4. If subdivisions of AB be bisected indefinitely, volume generated by polygon $OACDB$ approaches volume generated by sector OAB as a limit,* and area $ACDB \times \frac{1}{3} OE$ approaches area generated by arc $AB \times \frac{1}{3} R$ as a limit. (§§ 329, 330†)

5. Prove theorem by Theorem of Limits.

It is evident, as in § 588, that § 595 holds for any spherical sector.

Ex. 32. The angles of a spherical quadrilateral are 95° , 115° , 135° , and 110° , respectively. What is the area of the quadrilateral if the radius of the sphere is 7 inches?

* A rigorous proof of this is beyond the scope of the present treatise.

† See note, foot p. 284.

596. If V denotes the volume of a spherical sector, h the altitude of the zone which forms its base, R the radius of the sphere,

$$V = 2 \pi R h \times \frac{1}{3} R \text{ (§ 589) } = \frac{2}{3} \pi R^2 h.$$

597. Since a sphere may be regarded as a spherical sector whose base is the surface of the sphere,

The volume of a sphere is equal to the area of its surface multiplied by one-third its radius.

598. Let V denote the volume of a sphere, R its radius, and D its diameter.

$$\text{Then, } V = 4 \pi R^2 \times \frac{1}{3} R \text{ (§ 591) } = \frac{4}{3} \pi R^3.$$

That is, *the volume of a sphere is equal to the cube of its radius multiplied by $\frac{4}{3} \pi$.*

$$\text{Again, } V = \pi D^2 \times \frac{1}{6} D \text{ (§ 591) } = \frac{1}{6} \pi D^3.$$

That is, *the volume of a sphere is equal to the cube of its diameter multiplied by $\frac{1}{6} \pi$.*

599. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

(The proof is left to the pupil.)

600. *The volume of a spherical pyramid is equal to the area of its base multiplied by one-third the radius of the sphere.*

Let P denote volume of spherical pyramid, K the area of its base, R the radius of the sphere, n the number of sides of the base, s the sum of its $\angle$ s referred to a rt. $\angle$ as the unit of measure, T the area of a tri-rectangular Δ , T' the volume of a tri-rectangular pyramid, S the area of the surface of the sphere, and V its volume.

$$\text{Then, } \frac{P}{K} = \frac{[s - 2(n - 2)] \times T'}{[s - 2(n - 2)] \times T} = \frac{T'}{T}. \quad (\S\S 581, 582)$$

$$\text{Also, } \frac{V}{S} = \frac{8 T'}{8 T} = \frac{T'}{T}. \quad (\S 559)$$

$$\text{Then, } \frac{P}{K} = \frac{V}{S} = \frac{\frac{4}{3} \pi R^3}{4 \pi R^2} (\S \S 591, 598) = \frac{1}{3} R; \text{ and therefore}$$

$$P = K \times \frac{1}{3} R.$$

Ex. 33. A tank in the form of a cylinder of revolution is 2 meters in altitude, and the diameter of its base is 1 meter. How many gallons of water will it contain?

Ex. 34. A bar of iron in the form of a rectangular parallelepiped is 12 inches long, 8 inches wide, and 4 inches thick. The bar is melted and cast into a spherical ball. Find the radius of the ball.

Ex. 35. A brass rod is in the form of a cylinder of revolution, whose length is 4 feet and diameter of base 3 inches, capped at each end by a hemisphere whose diameter is 3 inches. The rod and hemispheres have a uniform thickness of $\frac{1}{8}$ inch. Find its weight. (One cubic foot of water weighs $62\frac{1}{2}$ pounds, and brass is 8.2 times as heavy as water.)

Ex. 36. A spherical metal shell is $\frac{1}{2}$ inch in thickness. The outside diameter of the shell is 10 inches. Find the amount of metal in the shell.

Ex. 37. The diameter of a sphere, the altitude and diameter of a cone of revolution and of a cylinder of revolution are each a inches. Their volumes are in what ratio? Their entire surfaces are in what ratio?

Ex. 38. Find ratio of the volumes of a sphere to its circumscribed cube.

Ex. 39. Find the ratio of the volume of a sphere to its circumscribed regular tetradron.

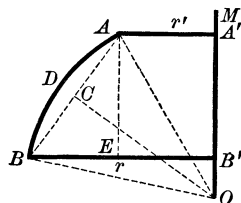
Ex. 40. The volume of a spherical wedge is 75. The volume of the sphere is 325. Find the angle of the base of the spherical wedge.

Ex. 41. Find the ratio of the volume of a sphere to its circumscribed cylinder of revolution. Could an oblique circular cylinder circumscribe the sphere? Why?

Ex. 42. Find the ratio of the volume of a sphere to the volume of the inscribed cube.

PROP. XXXIII. PROBLEM

601. *Given the radii of the bases, and the altitude, of a spherical segment, to find its volume.*



Given O the centre of arc ADB , lines AA' , $BB' \perp$ to diameter OM , $AA' = r'$, $BB' = r$, and $A'B' = h$.

Required to express volume of spherical segment generated by the revolution of $ADBB'A'$ about OM as an axis in terms of r , r' , h .

Solution. 1. Draw lines OA , OB , AB ; also, line $OC \perp AB$, and line $AE \perp BB'$; denote radius OA by R .

$$2. \quad \text{vol. } ADBB'A' = \text{vol. } ACBD + \text{vol. } ABB'A'. \quad (1)$$

$$3. \quad \text{vol. } ACDB = \text{vol. } OADB - \text{vol. } OAB.$$

$$4. \quad \text{By } \S 596, \text{ vol. } OADB = \frac{2}{3} \pi R^2 h.$$

$$5. \quad \text{By } \S 594, \quad \text{vol. } OAB = \text{area } AB \times \frac{1}{3} OC \\ = h \times 2 \pi OC \times \frac{1}{3} OC (\S 586) = \frac{2}{3} \pi \overline{OC}^2 h.$$

$$6. \quad \text{Then, vol. } ACDB = \frac{2}{3} \pi R^2 h - \frac{2}{3} \pi \overline{OC}^2 h = \frac{2}{3} \pi (R^2 - \overline{OC}^2) h.$$

$$7. \quad \text{By } \S 253, \quad R^2 - \overline{OC}^2 = \overline{AC}^2 = (\frac{1}{2} AB)^2 (?) = \frac{1}{4} \overline{AB}^2.$$

$$8. \quad \text{Then, vol. } ACDB = \frac{2}{3} \pi \times \frac{1}{4} \overline{AB}^2 \times h = \frac{1}{6} \pi \overline{AB}^2 h \\ = \frac{1}{6} \pi (\overline{BE}^2 + \overline{AE}^2) h = \frac{1}{6} \pi [(r - r')^2 + h^2] h.$$

$$9. \quad \text{Substitute in (1), by } \S 511, \text{ vol. } ADBB'A' \\ = \frac{1}{6} \pi [(r - r')^2 + h^2] h + \frac{1}{6} \pi (2 r^2 + 2 r'^2 + 2 r r') h \\ = \frac{1}{6} \pi (r^2 - 2 r r' + r'^2 + h^2 + 2 r^2 + 2 r'^2 + 2 r r') h \\ = \frac{1}{6} \pi (3 r^2 + 3 r'^2) h + \frac{1}{6} \pi h^3 = \frac{1}{2} \pi (r^2 + r'^2) h + \frac{1}{6} \pi h^3.$$

602. If r denotes radius of base, h the altitude, of a spherical segment of one base, its volume is

$$\frac{1}{2} \pi r^2 h + \frac{1}{6} \pi h^3.$$

Ex. 43. Find the volume of a cone of revolution inscribed in a sphere whose volume is 4500π . The axis of the cone is 15 and coincides with a fixed diameter. Can there be more than one such cone?

Ex. 44. A plane is drawn through the axis of a cone of revolution. The intersection of the plane and the cone is an equilateral triangle whose side is 12. Find the area of the surface of a sphere inscribed in the cone.

Ex. 45. A plane is passed through the axis of a cone of revolution. The intersection of the plane and the cone is an equilateral triangle. Find the ratio of the volume of the sphere inscribed in the cone to the volume of the cone.

Ex. 46. The legs of a right triangle are 18 and 12, respectively. The triangle revolves about the hypotenuse as an axis. Find the area of the surface generated.

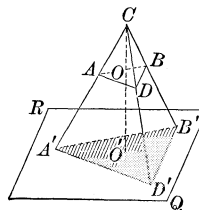
Ex. 47. A hollow iron column in the shape of a frustum of a cone of revolution is 10 feet long; one base is 10 inches in outer diameter, and 8 inches in inner, and the other base is 6 inches in outer diameter, and 4 inches in inner. The column is melted and formed into a bar in the shape of a rectangular parallelepiped, 4 inches wide and 2 inches thick. Find length of bar.

Ex. 48. A 10-pound spherical iron cannon ball is 3 inches in diameter. Find the diameter of a 20-pounder.

Ex. 49. Measure the circumference of a great circle of any sphere of known diameter. Using the measurement as given, compute the volume of the sphere.

Ex. 50. Given a semicircle of radius r , and a chord parallel to the diameter of the semicircle. The radii drawn from the middle point of the diameter to the extremities of the chord form an angle of 60° . The semicircle is revolved about its diameter as an axis. Find the volume of the solid generated by the segment between the chord and the arc.

Ex. 51. Given an equilateral triangle ABD , and a light placed at point C , equidistant from A , B and D , and at a distance 12 from plane ABD . A side of ABD is 8 and plane ABD is parallel to plane QR . $A'B'D'$ is the shadow of ABD upon QR , and CO' is perpendicular to $A'B'D'$, meeting ABD at O . The distance from C to QR is 64. Find area $A'B'D'$.



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